An Objective Programming Method of Optimal Control on Population System

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Abstract: On the basis of the present study on the issues of population system, further study on the issues of optimal control of population system is taken in this paper. An objective programming method of optimal control on population system is given in it. [Nature and Science, 2004,2(2):33-35]

Key words: population system; optimal control; objectively programming method

1 Preface

We should make clear the developing target of population system in formulate a long-term population policy, in another word to keep ideal population quantity and state in near future. After long-term population objective determined, how to steady transient into ideal population state and quantity in relatively short time, is an important theoretical question that needs to be solved.

The essential even unique control variable in control population developing is average women fertility or total fertility[1], i.e. the number of bearing every husband and wife do. Thus the solution to this problem should be how to control the changing law of women fertility level in the next tens to hundred years. Only these population quantity and state can transient into ideal state step by step.

It has reported that the method to solve this problem in present literature[1~3]. This paper put forward an objective programming method according to characters of optimal control on population system.

2 Discrete Model of Population System

Suppose what we study is the population is developing in one district. The district can be a province, a city, or whole nation, even the whole world. If we calculate certain year's population of that district in age, then we can make conclusion of that year whole population state distribution. Make $x_i(t)$ represent t year up to *i* full year of life but not to *i+1* full year of life total population, where *i* rounding-off number, *i.e.* *i=0, 1, 2, m, m* is population maximum age. That is to say, $x_i(t)$ is total population of life interval *[i, i+1]* age; $u_i(t)$ is population of *t year* full year of *i* age dead in *t* to *t+1* year divide population of *t year* full year of *i* age, which can be written as:

$$u_i(t) = \frac{x_i(t) - x_{i+1}(t+1)}{x_i(t)} \tag{1}$$

$$\begin{cases} x_1(t+1) = [1-u_0(t)]x_0(t) + g_0(t) \\ x_2(t+1) = [1-u_1(t)]x_1(t) + g_1(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_m(t+1) = [1-u_{m-1}(t)]x_{m-1}(t) + g_{m-1}(t) \end{cases} \tag{2}$$

Where $g_i(t)$ is *t* year *i* age migrate population, when people move in it is positive, move out it is negative.

Supposed from *t-1* to *t* year new birth baby number is $\psi(t)$, $\psi(t)$ can be calculated as follows[1][4]:

$$\psi(t) = \beta(t)\sum_{i=a_1}^{a_2} k_i(t)h_i(t)x_i(t) \tag{3}$$

Where $\beta(t)$ is *t* year average women fecundity; a_1 is the minimize child-bearing age; a_2 is the maximize child-bearing age; $k_i(t)$ is *t* year the proportion of full year of *i* year women in *i* age whole women; $h_i(t)$ is *t* year *i* age women's bearing model[5].

The practical meaning of *(3)* is clear, for $k_i(t)x_i(t)$ is *t* year whole population of full year of *i* age women, $\beta(t)h_i(t)$ indicate from *t-1* to *t* this year *i* age average women bear infant number, so $\beta(t)h_i(t)k_i(t)x_i(t)$ is *t* year whole full year of *i* age women bear infant number per year. When *i* vary from a_1 to a_2, sum that is *t year* age in $[a_1, a_2]$ whole women bear infant number per year, that is $\psi(t)$. These infants couldn't live until *t* year statistic time, some infant died from illness etc. The infant number can live till *t year* statistic time is $x_0(t)$, so $\psi(t)-x(t)$ is the infant number dead in $t-1$ to *t year*. Define infant death rate $u_{00}(t)$ as:

$$u_{00}(t)=\frac{\psi(t)-x_0(t)}{\psi(t)} \quad (4)$$

From which we can obtain a result:

$$x_0(t)=[1-u_{00}(t)]\psi(t) \quad (5)$$

Unite *(2), (3), (5),* we have a whole discrete simultaneous equation of population system. As follows:

$$\begin{cases} \psi(t)=\beta(t)\sum_{i=a_1}^{a_2}k_i(t)h_i(t)x_i(t) \\ x_0(t)=[1-u_{00}(t)]\psi(t) \\ x_1(t+1)=[1-u_0(t)]x_0(t)+g_0(t) \\ x_2(t+1)=[1-u_1(t)]x_1(t)+g_1(t) \\ \cdots\cdots \\ x_m(t+1)=[1-u_{m-1}(t)]x_{m-1}(t)+g_{m-1}(t) \end{cases} \quad (6)$$

(6) is a discrete simultaneous equation which time space with year, the simultaneous equation takes the impression of infant death rate into account.

Now let:

$$b_i(t)=[1-u_{00}(t)][1-u_0(t)]k_i(t)h_i(t)$$
$$i=a_1,a_1+1,a_1+2\cdots\cdots,a_2 \quad (7)$$

When lead-in vector and matrix:

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix} \qquad G(t)=\begin{bmatrix} g_0(t) \\ g_1(t) \\ \vdots \\ g_{m-1}(t) \end{bmatrix}$$

$$H(t)=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1-u_1(t) & 0 & \cdots & 0 \\ 0 & 1-u_2(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1-u_{m-1}(t) & 0 \end{bmatrix}$$

$$B(t)=(b_{ij}(t))_{m\times m}=\begin{bmatrix} 0 & \cdots & 0 & b_{a_1}(t) & \cdots & b_{a_2}(t) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We have

$$x(t+1)=H(t)x(t)+\beta(t)B(t)x(t)+G(t) \quad (8)$$

Where *H(t)* called population state transient matrix, *B(t)* called fertility matrix, *x(t)* called population state vector, *G(t)* called population migrate vector. For example:

$$x(0)=\{x_1(0),\quad x_2(0),\quad \cdots\cdots \quad x_m(0)\} \quad (9)$$

(9) is the former condition of *(8)*, united *(8)* and *(9)*, we can have the whole discrete model on population system developing process, that is:

$$\begin{cases} x(t+1)=H(t)x(t)+\beta(t)B(t)x(t)+G(t) \\ x(0)=\{x_1(0),\quad x_2(0),\quad \cdots\cdots \quad x_m(0)\} \end{cases} \quad (10)$$

From (10) we know, it's a discrete double line system. Average women fertility rate is a control variable, we can obtain the goal of control population state by means control $\beta(t)$.

3 An Objective Programming Method of Optimal Control on Population System

The regional population moderate question often called population moderate, objective people in demography. Once the target is made certain, the task of population control is to find a long-term policy to realize this long-term target. As our nation station shows in the next 100 years ideal and achievable population state is about total 7 billion, then our country population will rise as 0. If this goal can be accepted by society, the following task is found a long-term population policy in some sense, which will help our national population quantity and state lead to this ideal target in the next year. Concrete to say that we should set down a long-range control law about average women bearing rate, make total population near 7 billion but natural rise rate tend to 0 after several years simultaneously.

Any community cannot practice force life and death balanced population policy, it must explore a populace control policy which can be accepted by society, thus make population developing process gain its ends naturally. In the view of systematic engineering, this is a typical optimal control problem. To apply about optimal decision theory and method of systematic engineering, first ensure objective function of control process from quantity, then should scientific determine control variable should be meted of restraint condition. If we wish to meet certain conditions and maximum approach ideal population total and state from *0* to *T* time under realism total population, suppose *t* time *i* full year of life the positive, negative partial variable between population and *t* time ideal *i* full year of life are

$d_i^+(t)$ and $d_i^-(t)$, i.e.

$$d_i^+(t) = \begin{cases} x_i(t) - x_i^*(t) & x_i(t) > x_i^*(t) \\ 0 & x_i(t) \le x_i^*(t) \end{cases} \quad (11)$$

$$d_i^-(t) = \begin{cases} 0 & x_i(t) \ge x_i^*(t) \\ x_i^*(t) - x_i(t) & x_i(t) < x_i^*(t) \end{cases} \quad (12)$$

Where $x_i^*(t)$ is *t year i* full year of life ideal population.

So objective function of optimal control on population system can write as:

$$J(T) = \min_{\beta(t)} \sum_{t=0}^{T-1} \sum_{t=0}^{m} [d_i^+(t) + d_i^-(t)] \quad (13)$$

(13) should meet following restrict condition:

3.1 Balance constraint

t year i age population $x_i(t)$, *t year i* age ideal population $x_i^*(t)$ and *t year i* age population positive variable, *t year i* age negative partial variable should have such relations:

$$x_i(t) - x_i^*(t) - d_i^+(t) + d_i^-(t) = 0 \quad (14)$$

3.2 Birthrate constraint

Average women bearing rate *β(t)* cannot choose at will. *e.g.* *β(t)* must be more than or equal to 0, if we take into account the conditions accepted by social people, *β(t)* must be more than or equal to *1*, because it is not to be accepted if one couple bear no child policy. On the other hand *β(t)* should not too large, in fact it would unachievable if *β(t)* is larger than average maximum bearing rate. Therefore *β(t)* value often varies in a predefine interval, that is

$$1 \le \beta(t) \le \beta_c \quad (15)$$

Where β_c is average women maximum birthrate.

In addition, if we consider the varying velocity of population system is not too fast, in other word, we should ask difference in neighboring year average women birthrate less than or equal to some preset point ε.

$$\beta(t) - \beta(t+1) \le \varepsilon \quad (16)$$

$$\beta(t+1) - \beta(t) \le \varepsilon \quad (17)$$

3.3 Population peak value constraint

Wish in [0, T] time, factual populace total peak value N_{max} not more than definite value N_0 this restrict could be written:

$$N_{\max}(t) \le N_0 \quad (18)$$

3.4 Foster index constraint

When *(t)* varies with time, the changing of population age composition should be considered, especially present able-bodied person will be aging day by day. When they lost their ability to work, there must need enough young labor to foster elder and children. That means in the trend to ideal population, social foster index (the number of older and children fostered by per labor) cannot be too large, e.g. less than or equal to a definite value p_0

$$p(t) \le p_0 \quad (19)$$

Where *p(t)* is *t* time social foster index.

3.5 Aging index constraint

As the *β(t)* changing, there should notice the whole society population aging problem. To prevent populace aging problem, in control process ([0, T] time), aging index $\omega(t)$ (average population age ratio average expected life) should be less than or equal to a definite value ω_0, this requirement can be written:

$$\omega(t) \le \omega_0 \quad (20)$$

3.6 Non-negative constraint

According to population system characters, all decision variable(β (t), $d_i^+(t)$ and $d_i^-(t)$) should not be negative. So besides every constraint above, also have

$$d_i^+(t) \ge 0 \quad (21)$$

$$d_i^-(t) \ge 0 \quad (22)$$

What this paper proposes is objective model of optimal control on population system from (13) to (22) we formulate programming model. The method solves optimal tactics by means of these model called objective programming method.

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