Mathematical modeling of nonlinear thermomechanical processes in rods made of heat-resisting alloys

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Abstract. This paper is focused on development of a mathematical model, corresponding computing algorithms and a suite of applications in DELPHI high level object-oriented programming language for digital simulation and study of the thermomechanical state of AHB-300 heat-resistant alloy, with presence of local thermal insulation, heat exchange, temperature and axial forces with regard to jamming ends of the rod. With that, the coefficient of thermal expansion is the function of temperature field to the length of the rod of limited length.

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Introduction

Many bearing structural elements used in significant thermal fields in presence of axial forces are made of the AHB-300 heat-resistant alloy. The main property of this alloy is that the coefficient of thermal expansion is strongly dependent on temperature field along the length of the rod member under consideration [1]. Therefore, study of the thermomechanical state of rod-shaped bearing structural elements under simultaneous axial forces, with local insulation, heat exchange and temperature that may be constant, may vary along the local length of the rod linearly and quadrically, is of particular interest in many processes of ensuring thermal resistance of structural elements working in complex thermal and force fields [2]. Existing methods of studying stabilized thermomechanical state of rods of limited length does not make it possible to take into account the relationship between the coefficient of thermal expansion and the temperature field, operating conditions and fixation, and until now no mathematical model has been developed of rods stabilized thermomechanical state for the above mentioned operating conditions of a structural element. Naturally, relevant computing algorithms are also lacking, as well as methods and a suite of applications that enable comprehensive digital study of the above complex processes [3]. On this basis, the goal of this work is to develop a mathematical model of thermomechanical state of the rod with consideration of its operation, basing on energy principle in combination with the Finite Elements Method.

Let us describe the problem. Let there be a vertical rod of AHB-300 heat-resistant alloy with limited length L [cm]. Its area of cross section is constant along its length and is equal to S [cm^2]. The top end of the rod is rigidly clamped. Axial tensile force P [kg] is applied to the bottom end. Through the cross-section areas of the top and the bottom ends of the rod heat is exchanged with the environment. Schematically, the rod consists of three parts. The first part is $0 \le x \le \frac{L}{3}$. Its lateral surface is completely heat insulated. Through the cross-section area, with the coordinate $x = 0$, heat is exchanged with the environment. Thus the heat transfer coefficient h0 $\left[\frac{W}{cm^2 \cdot {}^\circ C}\right]$, and the ambient temperature is $T_{env0}\left|{}^\circ C\right|$. The next part of the rod is $\frac{L}{3} \le x \le \frac{2L}{3}$. For this local part, temperature is given that varies along the coordinate according to the parabolic law. i.e.,

$$T(x) = \frac{3120}{\ell^2}x^2 - \frac{3120}{\ell}x + 800 \quad , \quad \left(\frac{L}{3} \le x \le \frac{2L}{3}\right) \quad \text{or} \quad (0 \le x \le \ell) \qquad (1)$$

ℓ - The length of the portion where the temperature is set is $T(x)$. In this case, $\ell = \frac{2L}{3} - \frac{L}{3} = \frac{L}{3}$ (cm).

Next comes the last, third part $\left(\frac{2L}{3} \le x \le L\right)$. Its lateral surface is completely heat insulated, as well. Through the cross-section area of the bottom end of the rod with coordinate $x = L$, heat is also exchanged with the environment. Here heat transfer coefficient is $h_L\left[\frac{W}{cm^2 \cdot {}^\circ C}\right]$ and temperature of the environment is $T_{envL}\left|{}^\circ C\right|$ (Fig. 1 (a-b)).

It is necessary to determine the following:

a) temperature field along the length of the rod with regard to real operating conditions;

b) rod elongation from the temperature field and applied axial tensile force P.

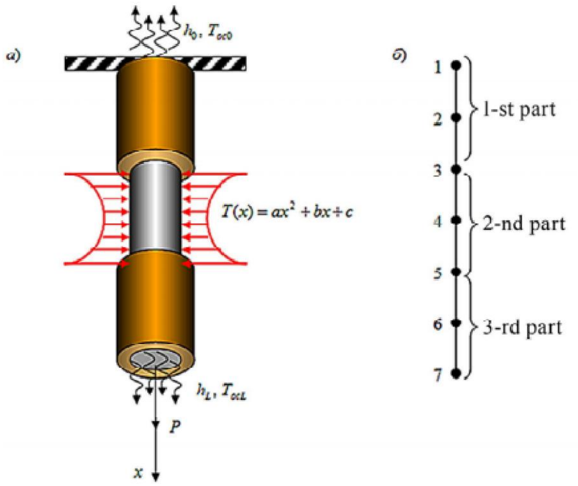

Figure 1. Calculation model of the problem in question.

a) problem definition scheme; b) scheme of discrete model.

Due to problem definition, temperature field in the second part $\left(\frac{L}{3}\le x\le\frac{2L}{3}\right)$ of the rod is defined as (1). Since the process in question is steady, the temperature field in the 1-st and the 3-rd parts will be a smooth curve. This curve in these parts we will approximate by a quadric curve passing through the three points. For example, for the1-st part we have that

$$T(x)=\varphi_1(x)\cdot T_1+\varphi_2(x)\cdot T_2+\varphi_3(x)\cdot T_3,\ _{\left(0\le x\le\frac{L}{3}\right)}. \qquad (2)$$

where $\varphi_1(x),\varphi_2(x),\varphi_3(x)$ are form functions for quadratic finite element with three nodes [4], in this case, for the 1-st part of the rod: T_1, T_2 u T_3 are the values of temperature in nodes 1, 2, and 3.

Similarly, for the 3-rd part we have

$$T(x)=\varphi_5(x)\cdot T_5+\varphi_6(x)\cdot T_6+\varphi_7(x)\cdot T_7,\ _{\left(\frac{2L}{3}\le x\le L\right)}. \qquad (3)$$

In the 2-nd expression $T_3=T\left(x=\frac{L}{3}\right)$.Besides, from (1) we have that $T_3=T(0)=800°C$. Also, in the expression (3) value T_5 is determined from (1) as $T_5=T(x=\ell)=800°C$. It will be necessary to determine T_1, T_2, T_6 and T_7.

To do so for the 1st part $0\le x\le\frac{L}{3}$, let us write the expression of total thermal energy basing on actual operating conditions

$$I=\int_{V_1}\frac{K_{xx}}{2}\left(\frac{\partial T}{\partial x}\right)^2dV+\int_{S_0}\frac{h_0}{2}(T-T_{env0})^2dS, \quad \left(0\le x\le\frac{L}{3}\right). \qquad (4)$$

Using (2), after integration from (4) we have

$$I_1=\frac{K_{xx}S}{2\ell}\left[\frac{7}{3}T_1^2-\frac{16}{3}T_1T_2+\frac{2}{3}T_1T_3-\frac{16}{3}T_2T_3+\frac{16}{3}T_2^2+\frac{7}{3}T_3^2\right]+\frac{h_0S_0}{2}\left[T_1-T_{env0}\right]^2 \qquad (5)$$

Similarly, for the 3-rd part $\left(\frac{2L}{3}\le x\le L\right)$ we have

$$I_2=\frac{K_{xx}S}{2\ell}\left[\frac{7}{3}T_5^2-\frac{16}{3}T_5T_6+\frac{2}{3}T_5T_7-\frac{16}{3}T_6T_7+\frac{16}{3}T_6^2+\frac{7}{3}T_7^2\right]+\frac{h_LS_L}{2}\left[T_7-T_{envL}\right]^2 \qquad (6)$$

where ℓ is the length of parts 1 and 3 in question. In both parts, $\ell=\frac{L}{3};\ S_0=S_L=S$. [5]

Now for the first part, given that $T_3=800°C$, by minimizing I_1 by T_1 and T_2, we obtain the following resolving system of linear algebraic equations

$$\begin{cases}\dfrac{\partial I_1}{\partial T_1}=0\\ \dfrac{\partial I_1}{\partial T_2}=0\end{cases}\Rightarrow\begin{cases}\dfrac{S\cdot K_{xx}}{2\ell}\left[\dfrac{14}{3}T_1-\dfrac{16}{3}T_2+\dfrac{2}{3}T_3\right]+Sh_0(T_1-T_{env0})=0\\ \dfrac{S\cdot K_{xx}}{2\ell}\left[-\dfrac{16}{3}T_1-\dfrac{16}{3}T_3+\dfrac{32}{3}T_2\right]=0\end{cases} \qquad (7)$$

In this system it should be noted that the sum of coefficients in square brackets is equal to zero. For example, for the 1-st equation of system (7) $\left[\frac{14}{3}-\frac{16}{3}+\frac{2}{3}\right]=0$. Similarly, for the second equation of this system, we have that $\left[-\frac{16}{3}-\frac{16}{3}+\frac{32}{3}\right]=0$. Now we can proceed to the third part $\left(\frac{2L}{3}\le x\le L\right)$ of the rod. From problem definition we know that here we have $T_5=800°C$. Then by minimizing I_2 by T_6 and T_7, we will obtain the following resolving system of linear equations

$$\begin{cases}\dfrac{\partial I_2}{\partial T_6}=0\\ \dfrac{\partial I_2}{\partial T_7}=0\end{cases}\Rightarrow\begin{cases}\dfrac{S\cdot K_{xx}}{2\ell}\left[-\dfrac{16}{3}T_5-\dfrac{16}{3}T_7+\dfrac{32}{3}T_6\right]=0\\ \dfrac{S\cdot K_{xx}}{2\ell}\left[\dfrac{2}{3}T_5-\dfrac{16}{3}T_6+\dfrac{14}{3}T_7\right]+S_L\cdot h_L(T_7-T_{envL})=0\end{cases} \qquad (8)$$

In this system the sum of coefficients in square brackets is also equal to zero. Now let us take the following for the input data:

$$K_{xx}=100\frac{W}{cm\cdot°C};\quad L=30\,cm\cdot$$

$$r=1cm;\ S=\pi\cdot r^2=\pi cm^2;$$

$$h_0=h_L=10\frac{W}{cm^2\cdot°C};\ T_{envO}=T_{envL}=40°C\cdot$$

With these initial data and taking into account existing boundary conditions from (7) and (8) we can find the following

Table 1.

Node numbers	1	2	3	4	5	6	7
Nodes coordinate	$x=0$	$x=\frac{L}{6}$	$x=\frac{L}{3}$	$x=\frac{L}{2}$	$x=\frac{2L}{3}$	$x=\frac{5L}{6}$	$x=L$
Temperature	$T_1=420°C$	$T_2=610°C$	$T_3=800°C$	$T_4=20°C$	$T_5=800°C$	$T_6=610°C$	$T_7=420°C$

Temperature field along the length of parts 1 and 3 is determined by relations (2) and (3).

Now we have to find rod elongation taking into account the relationship between the coefficient of thermal expansion α and the temperature field T(x). Such a full-scale dependence $\alpha = \alpha(T(x))$ for material AHB-300 we will take from [6]. In this paper, full-scale dependence $\alpha = \alpha(T)$ is given in the form of a graph. Using these data we can build the following table 2 of dependence $\alpha = \alpha(T)$.

Table 2.

$T°C$	$20°C$	$100°C$	$200°C$	$300°C$	$400°C$	$500°C$	$600°C$	$700°C$	$800°C$
$\alpha = \times 10^{-6}\,\frac{1}{°C}$	10.1	11.9	13.2	14.7	17	18.3	20.3	22	23.2

Now, using this table, let us build the distribution field $\alpha = \alpha(T(x))$ for the 1st part $\left(0 \le x \le \frac{L}{3}\right)$ of the rod. Comparing tables 1 and 2 we will find that

$$\begin{cases} \alpha_1 = \alpha(x=0) = \alpha(T(x=0)) = \alpha(420°C) = 17.2 \cdot 10^{-6}\,\frac{1}{°C} \\ \alpha_2 = \alpha\left(x = \frac{L}{6}\right) = \alpha\left(T\left(x = \frac{L}{6}\right)\right) = \alpha(610°C) = 20.4 \cdot 10^{-6}\,\frac{1}{°C} \\ \alpha_3 = \alpha\left(x = \frac{L}{3}\right) = \alpha\left(T\left(x = \frac{L}{3}\right)\right) = \alpha(800°C) = 23.2 \cdot 10^{-6}\,\frac{1}{°C}. \end{cases} \tag{9}$$

Next, using relations (9), the temperature field $\alpha = \alpha(x)$ in the range $\left(0 \le x \le \frac{L}{3}\right)$ of the rod can be represented as a quadric curve passing through three points $\left(x = 0;\ x = \frac{L}{6};\ x = \frac{L}{3}\right)$, i.e.

$$\alpha(x) = \varphi_1(x)\cdot\alpha_1 + \varphi_2(x)\cdot\alpha_2 + \varphi_3(x)\cdot\alpha_3,\ \left(0 \le x \le \frac{L}{3}\right) \tag{10}$$

Then elongation of first part $\left(0 \le x \le \frac{L}{3}\right)$ of the rod from temperature field (2) is

$$\Delta l_{T1} = \int_0^{\frac{L}{3}} \alpha(x)\cdot T(x)dx =$$
$$= \int_0^1 [\varphi_1(x)\cdot\alpha_1 + \varphi_2(x)\cdot\alpha_2 + \varphi_3(x)\cdot\alpha_3]\cdot[\varphi_1(x)\cdot T_1 + \varphi_2(x)\cdot T_2 + \varphi_3(x)T_3]dx =$$
$$= \int_0^1 \big[\varphi_1^2(x)\alpha_1T_1 + \varphi_1(x)\varphi_2(x)\alpha_1T_2 + \varphi_1(x)\varphi_3(x)\alpha_1T_3 + \varphi_1(x)\varphi_2(x)\alpha_2T_1 + \varphi_2^2(x)\alpha_2T_2 +$$
$$+ \varphi_2(x)\varphi_3(x)\alpha_2T_3 + \varphi_1(x)\varphi_3(x)\alpha_3T_1 + \varphi_2(x)\varphi_3(x)\alpha_3T_2 + \varphi_3^2(x)\alpha_3T_3\big]dx = |with\ \ 1 = 10\,cm|$$
$$= 0.12593386 \text{ cm}.$$

For comparison, let us calculate elongation of the same part without dependence between the coefficient of thermal expansion and temperature [7]. Thus, let us calculate the elongation of part $\left(0 \le x \le \frac{L}{3}\right)$ of the rod with $\alpha = const = 10.1\cdot 10^{-6}\,\frac{1}{°C}$. It is equal to

$$\Delta l_{T1}(\alpha = const) = \alpha\int_0^{1_1} T(x)dx = \alpha\left[\frac{1}{6}T_1 + \frac{21}{3}T_2 + \frac{1}{6}T_3\right] = \frac{\alpha 1}{6}\cdot(T_1 + 4T_2 + T_3) =$$
$$= \frac{10.1\cdot 10^{-6}\cdot 10}{6}\cdot(420 + 4\cdot 610 + 800) = 0.06161\,cm.$$

These calculations show that when the relationship is considered between the coefficient of thermal expansion and temperature field, the value of elongation of the part $\left(0 \le x \le \frac{L}{3}\right)$ of the rod will be more 2.04 times more than when $\alpha = \alpha(T)$ is not taken into account. Now let us calculate elongation of the 2-nd, i.e., the middle part $\left(\frac{L}{3} \le x \le \frac{2L}{3}\right)$. Here temperature field is shown as (1). Therefore, using Table 2, we can find elongation of this part [8].

$$\Delta\ell_{T2} = \int_{\frac{L}{3}_1}^{\frac{2L}{3}} \alpha(T(x))\cdot T(x)dx =$$
$$= \int_0^{\ell} [\varphi_i(x)\cdot\alpha_i + \varphi_j(x)\cdot\alpha_j + \varphi_k(x)\cdot\alpha_k]\cdot\left[\frac{3120}{\ell^2}x^2 - \frac{3120}{\ell}x + 800\right]dx =$$
$$= \int_0^1 \left[\frac{1}{1^2}(1^2 - 31x + 2x^2)\cdot 23.2\cdot 10^{-6} + \frac{1}{1^2}(41x - 4x^2)\cdot 10.1\cdot 10^{-6} + \frac{1}{1^2}(2x^2 - 1x)\cdot 23.2\cdot 10^{-6}\right]\times$$
$$\times\left[\frac{3120}{1^2}x^2 - \frac{3120}{1}x + 800\right]dx = |with\ \ 1 = 10\,cm| = 0.04959\,cm.$$

With constant $\alpha = 10.1\cdot 10^{-6}\,\frac{1}{°C}$, i.e., without dependence between the coefficient of thermal expansion on the temperature, values of elongation of the second part $\left(\frac{L}{3} \le x \le \frac{2L}{3}\right)$ of the rod would be equal to

$$\Delta l_{T2}\left(\alpha = const = 10.1\cdot 10^{-6}\,\frac{1}{°C}\right) = \alpha\int_0^1\left[\frac{3120}{1^2}x^2 - \frac{3120}{1}x + 800\right]dx =$$
$$= \alpha\left[\frac{3120}{1^2}\cdot\frac{x^3}{3} - \frac{3120}{1}\cdot\frac{x^2}{2} + 800x\right]_0^{1=10cm} = 0.02828\,cm.$$

This shows that when the dependence $\alpha = \alpha(T)$ is taken into account, elongation of the second part $\left(\frac{L}{3} \le x \le \frac{2L}{3}\right)$ of the rod is more by 75.35% than when it is not taken into account.

Due to the symmetry condition with respect to the middle part of the rod in question, elongation of the 3-rd part $\left(\frac{2L}{3} \le x \le L\right)$ will be equal to that of the 1-st part. Then $\Delta l_{T3} = \Delta l_{T1} = 0.12593386\,cm$. Thus, total elongation of the rod in question from the temperature field $T = T(x)$, $(0 \le x \le L)$ will be $\Delta l_T = \Delta l_{T1} + \Delta l_{T2} + \Delta l_{T3} = 0.30145772\,cm$.

Elongation of the rod in question from the axial tensile force P = 1000 kg applied to the bottom end of the rod, according to Hooke's law [9] is $\Delta l_P = \frac{PL}{ES} = \frac{1000\cdot 30}{2.1\cdot 10^6\cdot 3.14} = 0.00045496\,cm$. Then the total elongation of the rod will be $\Delta l = \Delta l_T + \Delta l_P = 0.30191268\,cm$.

With $h_0 = 5\frac{W}{cm^2\cdot°C};\ T_{env} = 30°C$ and $h_L = 10\frac{W}{cm^2\cdot°C};\ T_{envL} = 40°C$, temperature value T_1, T_2,

T_3 will be, and will be $T_1 = 543{,}3°C$, $T_2 = 671{,}66°C$ and $T_3 = 800°C$. Then the temperature field in the first part $\left(0 \le x \le \frac{L}{3}\right)$ of the rod has the following form:

$$T(x) = 543{,}3 \cdot \varphi_i(x) + 671{,}66 \cdot \varphi_j(x) + 800 \cdot \varphi_k(x), \quad (0 \le x \le L).$$

According to Table 2, we have that

$$\alpha_i = \alpha(x=0) = 19 \cdot 10^{-6} \frac{1}{°C}; \; \alpha_j = \alpha\left(x = \frac{L}{6}\right) = 21{,}3 \cdot 10^{-6} \frac{1}{°C}; \; \alpha_k = \alpha\left(x = \frac{L}{3}\right) = 23{,}2 \cdot 10^{-6} \frac{1}{°C}$$

With these values α, its distribution field in the first section can be represented as:

$$\alpha = \alpha(x) = 19 \cdot 10^{-6} \cdot \varphi_i(x) + 21{,}3 \cdot 10^{-6} \cdot \varphi_j(x) + 23{,}2 \cdot 10^{-6} \cdot \varphi_k(x)$$

$$\left(0 \le x \le \frac{L}{3}\right).$$

In this case, elongation of first part of the rod from temperature field (2) is defines as follows:

$$\Delta l_{T1} = \int_0^{\frac{L}{3}} \alpha(x) \cdot T(x) dx =$$

$$= \int_0^{10} \left[19 \cdot 10^{-6} \varphi_i(x) + 21.3 \cdot 10^{-6} \varphi_j(x) + 23.2 \cdot 10^{-6} \varphi_k(x)\right] \cdot \left[543.3 \varphi_i(x) + 671.66 \varphi_j(x) + 800 \varphi_k(x)\right] dx =$$

$$= 0.0137636 + 0.0085 - 0.005 + 0.0077 + 0.0763 + 0.01136 - 0.0042 + 0.01039 + 0.024746 =$$

$$= 0.1435596\, cm.$$

This shows that with $h_0 = 5 \frac{W}{cm^2 \cdot °C}$; $T_{env0} = 30°C$, elongation of the first part $\left(0 \le x \le \frac{L}{3}\right)$ of the rod is greater by ≈14% than with $h_0 = 10 \frac{W}{cm^2 \cdot °C}$; $T_{env0} = 40°C$. This phenomenon is motivated by the fact that with $h_0 = 5 \frac{W}{cm^2 \cdot °C}$; $T_{env0} = 30°C$ heat losses will be less than with $h_0 = 10 \frac{Bm}{cм^2 \cdot °C}$; $T_{oc0} = 40°C$. Elongation of other parts will be the same as those in case of $h_0 = 10 \frac{W}{cm^2 \cdot °C}$; $T_{env0} = 40°C$ [10].

Conclusion

Thus, the developed computing algorithm and the method make it possible to determine regularities of thermomechanical processes in rods with limited length in complex thermal and stress fields.

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References

1. Segerlind, L., 1979. Application of the method. Mir, p. 392.
2. Casting Design and Performance, 2009. ASM International, p. 264.
3. Brenner, S.C., 2008. The Mathematical Theory of Finite Element Methods. Springer Science & Business Media, 15: 397.
4. Zienkiewicz, O., 1975. The Finite Element Method in engineering. Mir, p. 541.
5. Ciarlet, P.G., 1978. The Finite Element Method for Elliptic Problems, North-Holland: Amsterdam, p. 530.
6. Khimushin, F.F., 1969. Heat-resistant steels and alloys. 2nd revised and updated edition. Metallurgy, p. 749.
7. Kudaykulov, A.K., 2009. Mathematical modeling of applied problems of heat propagation in one-dimensional structural elements. Turkestan: Baiterek, p. 168.
8. Kenzhegulov, B., I. Ishutina and N.B. Kenzegulov, 2013. Computer modeling at teaching students of it technology. Forming and qualitative development of modern educational systems. In the Proceedings of the LXIV International Research and Practice Conference and the III stage of the Championship in Pedagogical Sciences, London, pp: 65-67.
9. Timoshenko, S.P. and J.N. Goodyear, 1975. Theory of Elasticity.Nauka, p. 575.
10. Kudaykulov, A.K., A.N. Myrzasheva and B.Z. Kenzhegulov, 2009. Mathematical modeling of thermo mechanical processes in pivotal element of the designs made from thermal stable infusible alloys. // III congress of the world mathematical society os turkic countries, Kazakh National University named after Al-Farabi, Almaty: p. 111.