Some results on the dihedral homology of Banach algebras

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Abstract: In this paper we describe a technique for calculation of the Banach dihedral homology groups of Banach algebras, to establish the basic properties of this technique and to apply it to some classes of algebras. The technique involves some concepts of relative dihedral homology of the unital Banach algebra with involution. Therefore, we define the free involutive resolution of Banach algebra and given some theorems on the relative dihedral homology of the unital Banach algebra.

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1. Introduction

Many interest has been attached in recent gears the computation of Simplicial, cyclic, dihedral of (co)homology groups Banach algebra and in particular C*-algebra (see [1],[7], [8]).

Firstly, we recall some definition and facts needed in the sequel (see [3],[5],[6]).

Let A be a unital Banach algebra with involution (a unital Banach algebra is a Banach algebra with unit e such that $\|e\|=1$). We denote by $C_n(A), \quad n=0,1,...(n+1)$ fold projective tensor power as $A^{\otimes(n+1)}=A\otimes...\otimes A$ of *A*; we shall call the elements of these Banach space n-dimensional chains. We let $t_n : C_n(A)\to C_{n-1}(A), \quad n=0,1,2,...,$ denote the operator given by

$$d_n(a_0\otimes a_1\otimes...\otimes a_n)=$$
$$\sum_{i=0}^{n}(-1)^i(a_0\otimes...\otimes a_ia_{i+1}\otimes...\otimes a_{n+1})$$
$$+(-1)^{n+1}(a_{n+1}a_0\otimes...\otimes a_n). \tag{1.1}$$

It is well known $b_{n-1}b_n=0,$ for all $n\in N,$ is clearly equivalent to $\operatorname{Im} d_{n+1}\subset \ker d_n$. The element of $\operatorname{Im} d_{n+1}$ called *n*-boundaries and the elements of $\ker d_n$ are called *n*-cycles. The complex $C(A)=(C_\bullet(A),d_\bullet)$ is a chain complex as $C(A)$:

$$0\leftarrow C_0(A)\xleftarrow{d_0}...C_n(A)\xleftarrow{dn}C_{n+1}(A)\leftarrow... \tag{1.2}$$

This complex is called the Hochschild (simplicial) complex and it is homology is called the Hochschild Homology $H_n(A)=H_n(A,A)=\dfrac{Ker\ b_n}{\operatorname{Im}\ b^{n+1}}$.

We let $t_n : C_n(A)\to C_n(A), n=0,1,...$ denote the operator given by

$$t_n(a_0\otimes a_1\otimes...\otimes a_n)=(-1)^n(a_n\otimes a_0\otimes...\otimes a_{n-1})$$

And we set $t_0=id$. We let $C_n(A)$ denote the quotient space of $C_n(A)$ modulo the closure of the linear span of elements of the form $x\to t_nx$ where $n=0,1,...$. Note that, from [1], Im $(1-t_n)$ is closed in $C_n(A)$ and $CC_n(A)=C_n(A)/\operatorname{Im}(1-t_n)$. Thus we obtain a quotient complex $CC_*(A).$ of complex $CC(A)$. The homology of $CC_*(A)$, denoted by $HC_n(A)$ is called the *n*-dimensional Banach cyclic homology group of A we let $r_n : C_n(A)\to C_n(A), n=0,1,...$ denote the operator given by the formula:

$$r_n(a_0\otimes...\otimes a_n)=(-1)^{\frac{n(n+1)}{2}}\in a_0^*\otimes a_n^*\otimes...\otimes a_1^*, \quad \in=\pm1 \tag{1.3}$$

Where * is an involution on *A*. Note that $\operatorname{Im}(id_{t_n(A)}=1-t_n)$ is closed in $C_n(A).$

The quotient complex

$$CD_n(A)=\frac{C_n(A)}{\operatorname{Im}(1-t_n)+\operatorname{Im}(1-r_n)}$$

of a complex $C_n(A)$. The n-dimensional homology of $CD_n(A)$ denoted by $HD_n(A)$. is called n-dimensional dihedral homology group of a unital Banach algebra A.

2. Results

Let A and B be commutative Banach algebras with unit. Let $f{:}A \to B$ be an involutive Banach algebras homomorphism. We define a free involutive resolution of Banach algebras B over the homomorphism-ism $f : A \xrightarrow{i} R \xrightarrow{\pi} B$, where i is an inclusion map and π is a quasi-isomorphism, we define the relative dihedral homology;

$${}^{\varepsilon}HD_*(A \xrightarrow{f} B) = H_*(R/(A+[R,R]+\operatorname{Im}(1-r^{\varepsilon}))$$

where *[R,R]* is the commutant of Banach algebra *R*, r^{ε} is an involution on *R*, and study its properties operator algebra. Let f be a homomorphism of involutive Banach algebras *A* and *B* over a field *K* with characteristic zero. Let R_f^B be a free resolution of Banach algebra *B* over f and for $r_1, r_2 \in R_f^B$, let $[r_1, r_2] = r_1 r_2 - (-1)^{|r_1||r_2|} r_2 r_1$ where $|r_i| = \deg r_i$, $i = 1,2.$

Let $C = [R_f^B, R_f^B]$ be the linear space generated by $[r_1, r_2]$, $r_1, r_2 \in R_f^B$. We construct the complex $(C = [R_f^B, R_f^B] + \operatorname{Im}(1-r^{\varepsilon}))$ where

$$r^{\varepsilon}(P) = \varepsilon(-1)^{|P|(|P|-1)/2} P^*,$$

and * is an involution on R_f^B, $\varepsilon = \pm 1$. Clearly, from the definition of R_f^B, that *[Im (1-r^ε)]* is a subcomplex of R_f^B, we have

$$\begin{aligned}
\partial[r_1 r_2] &= r_1 r_2 - (-1)^{|r_1||r_2|} r_1 r_2 \\
&= \partial r_1 r_2 + (-1)^{|r_1|} r_1 \partial r_2 - (-1)^{|r_1||r_2|}(\partial r_2 r_1 + (-1)^{|r_2|} r_2 \partial r_1) \\
&= \partial r_1 r_2 - (-1)^{|r_2|(|r_1|+1)} r_2 \partial r_1 + (-1)^{|r_1|}(r_1 \partial r_2 - (-1)^{|r_1|(|r_2|+1)} \partial r_2 r_1) \\
&= [\partial r_1 r_2] + (-1)^{|r_1|}[r_1, \partial r_2], \qquad |\partial r_i| = |r_i| - 1, \qquad i = 1,2.
\end{aligned}$$

(2.1)

Then $[R_f^B, R_f^B]$ is subcomplex in R_f^B, therefore, the chain complex of *K-module* which is $([R_f^B, R_f^B] + \operatorname{Im}(1-r^{\varepsilon}))$ is a subcomplex of R_f^B.

Definition 2.1: Let $f : A \to B$ be *F-Banach* algebras *(char K=0*) homomorphism, R_f^B, be a free resolution of Banach algebra *B* over *f*. Then the relative dihedral homology is defined by:

$${}^{\varepsilon}HD_*(A \xrightarrow{f} B) = H_*\left(\frac{R_f^B}{(A+[R,R]+\operatorname{Im}(1-r^{\varepsilon}))}\right) \tag{2.2}$$

Definition 2.2: The *F-Banach algebra* A<t> generated by the elements $a_0 t a_1 t..t a_n$, $n \geq 0$, can be considered as differential graded Banach algebras by requiring that the morphism $A \to A\langle t \rangle$ is a morphism of involutaive differential graded Banach algebras and $\deg\ t = 1$, $\partial t = 0$ and $t^* = t$.

Lemma 2.3: The Banach algebras A<t> is splitable. It is a free Banach algebras resolution of the Banach algebras B=0 over the homorphism $A \to 0$.

Proof. Define the following chain complex

$$A \xleftarrow{\partial} AtA \xleftarrow{\partial} AtAt \xleftarrow{\partial} \ldots At\ldots tA \xleftarrow{\partial} \ldots, \tag{2.3}$$

where $At\ldots tA$ *(n-times*) is a *K*-module and the boundary operator ∂ is given by:

$$\begin{aligned}
\partial(a_0 t a_1 t..t a_{n-1} t a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 t a_1 t \ldots t a_i (\partial t) a_{i+1} t \ldots t a_n \\
&= \sum_{i=0}^{n-1} (-1)^i a_0 t a_1 t \ldots t(a_i a_{i+1}) t \ldots t a_n
\end{aligned} \tag{2.4}$$

Note that the differential ∂ in A<t> is equivalent to the operator $\delta_n^{`} : C_n(A) \to C_{n-1}(A)$ (see[2],[10]), defined by

$$\delta_n^{`}(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n. \tag{2.5}$$

From [9] the complex $(C_n(A), \delta_n^{`})$ is splitable and so the complex A<t> is also splitable, that is, $H_*(A\langle t\rangle) = 0$. Then, a Banach algebra A<t> is free resolution of the Banach algebra $B=0$ over the homomorphism $A \to 0$.

Lemma 2.4: The complex ($A\langle t\rangle/[A, A\langle t\rangle]$) is standard simplicial complex.

Proof. Let the complex ($A\langle t\rangle/[A, A\langle t\rangle]$), it's generated by the elements $a_0ta_1t...ta_{n-1}t$, since

$$a_0ta_1t...ta_n = a_na_0ta_1t...ta_{n-1}t, \quad (\mathrm{mod}\,[A, A<t>])$$

. The action of the differential ∂ on the complex ($A\langle t\rangle/[A, A\langle t\rangle]$) is given by

$$\partial(a_0ta_1t..ta_{n-1}ta_n) = \sum_{i=0}^{n-1}(-1)^i a_0ta_1t....t(a_ia_{i+1})t....ta_n + (-1)^n a_na_0ta_1t....a_{n-1}t \tag{2.6}$$

Consider the complex:

$$A \xleftarrow{id} A \xleftarrow{\delta} A^{\otimes 2} \xleftarrow{\delta} ...A^{\otimes n} \xleftarrow{\delta} ..., \tag{2.7}$$

where δ is the differential in the standard *Hochschild* complex. Since the space ($A\langle t\rangle/[A, A\langle t\rangle]_{n+1}$) identifies with the space;

$$A^{\otimes n+1} : a_0ta_1....ta_nt \to a_0 \otimes a_1 \otimes .... \otimes a_n, \tag{2.8}$$

and the differential in ($A\langle t\rangle/[A, A\langle t\rangle]$) identifies with the differential in the standard *Hochschild* complex.

Theorem 2.5: Let A be a unital Banach algebra with involution. Then $^{\varepsilon}HD_i(A \xrightarrow{f} B) = {}^{\varepsilon}HD_{i-1}(A)$, where $^{\varepsilon}HD_i(A)$ is the dihedral homology of *F*- Banach algebras *(char K=0)*.

Proof: Consider the factor complex ($A\langle t\rangle/[A, A\langle t\rangle] + \mathrm{Im}(1-r^{\varepsilon})$) such that;

$$\begin{aligned} a_0ta_1t...ta_{n-1}t &= (-1)^{n(n-1)/2}\ \varepsilon\ ta_n^*ta_{n-1}^*...ta_1^* \\ &= (-1)^{n(n-1)/2}\ \varepsilon\ ta_0^*ta_n^*...ta_1^*t, \end{aligned} \tag{2.9}$$

where $\varepsilon=\pm 1$,

$$\deg a_0ta_1t...ta_{n-1}t = n, \quad \deg(a_n^*) = 0,$$
$$\deg a_0ta_1t...ta_nt = n+1.$$

The dihedral homology of $A<t>$ is the dihedral homology of the complex $(A<t>/[A<t>, A<t>] + \mathrm{Im}(1-r^{\varepsilon}))$.

By factor $A<t>$ first by the subcomplex $A \leftarrow 0 \leftarrow 0 \leftarrow \cdots$ and second by the **Subcopmlex** $(A<t>/[A<t>, A<t>] + \mathrm{Im}(1-r^{\varepsilon}))$ **we get a homomorphism** $^{\varepsilon}CD_*(A \to 0) \to {}^{\varepsilon}CD_{*-1}(A)$, which induces an isomorphism of the dihedral homology groups $^{\varepsilon}HD_*(A \to 0) \to {}^{\varepsilon}HD_{*-1}(A)$.

Theorem 2.6: Let $f : A \to B$ be a homomorphism of a commutative Banach algebras over a field K (*char* $K=0$). Then the relative dihedral homology $^{\varepsilon}HD_i(A \xrightarrow{f} 0)$, does not depends on the choice of the resolution.

Proof: The homomorphism f induces homomorphism of chain complexes

$$f_* \ :\ {}^{\varepsilon}CD_*(A) \to {}^{\varepsilon}CD_*(B) \tag{2.10}$$

where $^{\varepsilon}CD_*(A)$ is a dihedral complex. Consider the diagram

$$\begin{array}{ccc} & & R_f^B \\ i & \nearrow & \downarrow \pi \\ A & \xrightarrow{f} & B \end{array} \tag{2.11}$$

Where R_f^B, is defined above, i is an inclusion map. Since

$$H_i(R_f^B) = \begin{cases} B, i = 0 \\ 0, i > 0, \end{cases} \tag{2.12}$$

Then the isomorphism $\pi_* : {}^{\varepsilon}CD_*(R_f^B) \to {}^{\varepsilon}CD_*(B)$ induces an isomorphism of the homology of these omplexes. Since

$$^{\varepsilon}HD_i(A \xrightarrow{f} B) \to {}^{\varepsilon}HD_i(A \xrightarrow{gof} C) \to {}^{\varepsilon}HD_i(A \xrightarrow{g} C) \to {}^{\varepsilon}HD_{i-1}(A \xrightarrow{f} B) \to ... \tag{2.13}$$

where $i_* : {}^{\varepsilon}CD_*(A) \to {}^{\varepsilon}CD_*(R_f^B)$ is an inclusion, then $M(i_*) \approx [{}^{\varepsilon}CD_*(R_f^B)/{}^{\varepsilon}CD_*(A)]$ where $M(i_*)$ is a cone of i,

$$\begin{array}{ccc} & & {}^{\varepsilon}CD_*(R_f^B) \\ i_* & \nearrow & \downarrow \pi_* \\ {}^{\varepsilon}CD_*(A) & \xrightarrow{f_*} & {}^{\varepsilon}CD_*(B) \end{array} \tag{2.14}$$

(see [12]).

The symbol $\approx$ denotes a quasi-isomorphism. It is clear, that the following diagram is commutative and hence $M(f_*) \approx {}^{\varepsilon}CD_*(R_f^B)/{}^{\varepsilon}CD_*(A)$.

Following ([4],[7]), we have

$[CC_*(R_f^B)/\ CC_*(A) \approx R_f^B / A + [R_f^B, R_f^B]]$,

where CC_* is the Conne`s cyclic complex, and by using the spectral sequence $E_{ij}^2 = {}^{\varepsilon}H_*(Z/2, H_*(R_f^B)) = {}^{\varepsilon}HD_{i+j}(R_f^B)$, we have

$$
{}^{\varepsilon}CD_*(R_f^B) / {}^{\varepsilon}CD_*(A) \approx R_f^B / A + [R_f^B, R_f^B] + \mathrm{Im}(1 - r^{\varepsilon}) \qquad (2.15)
$$

so,

$$
M(f_*) \approx (R_f^B / A + [R_f^B, R_f^B] + \mathrm{Im}(1 - r^{\varepsilon})).
$$

Then ${}^{\varepsilon}HD_i(A \xrightarrow{f} B)$ does not depend on the choice of R_f^B.

Theorem 2.7: Let *A, B* and *C* are involutive Banach algebra. Then the following sequence $A \xrightarrow{f} B \xrightarrow{g} C$ induces the long exact sequence of relative dihedral homology;

$$
{}^{\varepsilon}HD_i(A \xrightarrow{f} B) \to {}^{\varepsilon}HD_i(A \xrightarrow{gof} C) \to {}^{\varepsilon}HD_i(A \xrightarrow{g} C) \to {}^{\varepsilon}HD_{i-1}(A \xrightarrow{f} B) \to \dots \qquad (2.16)
$$

Proof: From theorem (2.6), we have been proved that any homomorphism $f: A \to B$ of involutive algebra in an arbitrary category is equivalent to an inclusion $i: A \to R_f^B$. Then, for a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of involutive Banach algebra, we have the following complex

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[dr, "f"'] & R_f^B \arrow[r, equal] \arrow[d, equal] & B \arrow[r, "i`"] \arrow[dr, "g"'] & R_f^B \arrow[d, equal] \\
& B & & C
\end{tikzcd} \qquad (2.17)
$$

Consider the following sequence of mapping cones

$$
0 \to M(i_*) \to M(i`_*) \to M(i_* o i`_*) \to 0 \qquad (2.18)
$$

In general, the sequence (2.18) is not exact and the composition of two morphism will be zero. However, the cone over morphism $M(i_*) \to M(i`_*)$, is canonically homotopy equivalent to $M(i_* \circ i`_*)$.

So we get the following exact sequence of the relative dihedral homology

$$
{}^{\varepsilon}HD_i(A \xrightarrow{f} B) \to {}^{\varepsilon}HD_i(A \xrightarrow{gof} C) \to {}^{\varepsilon}HD_i(A \xrightarrow{g} C) \to {}^{\varepsilon}HD_{i-1}(A \xrightarrow{f} B) \to \dots \qquad \textbf{(2.19)}
$$

Example 2.8: Let A be *F-Banach algebra* (*char K=0*) and M is A-bimodule. For a chain complex $U_\bullet$ of involutive Banach algebra, consider the complex $S^n(A, U_\bullet) = A \otimes_{A \otimes A^{op}} U_\bullet^{\otimes(k+1)}$. If we act on $S^n(A, U_\bullet)$ by the dihedral group D_{n+1} of order $2(n+1)$ by means :

$$
t_n(u_0 \otimes \dots \otimes u_n) = (-1)^{\mu} u_n \otimes v_0 \otimes \dots \otimes u_{n-1},
$$

$$
{}^{\alpha}r_n(u_0 \otimes \dots \otimes u_n) = (-1)^{\theta} \alpha u^*_n \otimes \dots \otimes u^*_1 \otimes u^*_0,
$$

$$
= (-1)^{n(n+1)/2} \varepsilon t a_0^* t a_n^* \dots t a_1^* t, \quad \alpha = \pm 1
$$

where

$$
\mu = (\deg \mathrm{p_n})(\sum_{i=0}^{n-1} \deg \mathrm{p_i}) = (n + \sum_{i=0}^{n} \deg \mathrm{p_i})(n + \sum_{i=0}^{n} \deg \mathrm{p_i} - 1)/2.
$$

If $U_\bullet$ is free involutive resolution of a Banach algebra *A,* then the complex $S^n(A, U_\bullet)$ can be considered by the complex $S^n(A, M)$.

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