Discussing the Existence of the Solutions and Their Dynamics of some Difference Equations

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Abstract: In this paper we care about the existence and study some qualitative properties of solutions to the following rational nonlinear difference equation $x_{n+1}=\frac{x_{n-(3k+2)}}{b+cx_{n-k}x_{n-(2k+1)}x_{n-(3k+2)}}$, $n=0,1,...,$ where b and c are real numbers, k is a non-negative integer number and the initial conditions $x_{-3k-2}, x_{-3k-1},..., x_{-1}, x_0$ are arbitrary non-negative real numbers. Also, we derive the solutions of some special cases of the equation under consideration.

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1. Introduction

Our aim in this paper is to investigate the dynamics of the solutions to the following difference equation

$$x_{n+1}=\frac{x_{n-(3k+2)}}{b+cx_{n-k}x_{n-(2k+1)}x_{n-(3k+2)}},\quad n=0,1,..., \tag{1}$$

where b and c are real numbers, k is a non negative integer number and the initial conditions $x_{-3k-2}, x_{-3k-1},..., x_{-1}, x_0$ are arbitrary non-negative real numbers. Also, we obtain the solutions of some special cases of Eq.(1).

The study of Difference Equations has been growing continuously for the last decades. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear rational difference equations. The study of this kind of equations is quite challenging and rewarding and is still in its infancy. The nonlinear rational difference equations are of paramount importance in their own right, and furthermore we believe that these results about such equations over prototypes for the development of the basic theory of the global behavior of nonlinear rational difference equations.

Some results on rational difference equations and systems of difference equations can be found in refs. [1-23].

Let I be some intervals of real numbers and let

$$f: I^{k+1}\to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1},..., x_0\in I$, the difference equation

$$x_{n+1}=f(x_n, x_{n-1},..., x_{n-k}),\quad n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Periodicity)
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1}=\sum_{i=0}^{k}\frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}}y_{n-i}. \tag{3}$$

Theorem A [15]:
Assume that $p_i\in R, i=1,2,...,k$ and $k\in\{0,1,2,...\}$. Then

$$\sum_{i=1}^{k}|p_i|<1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \ n = 0,1,\ldots.$$

2. Local Stability and Boundedness of Eq.(1)

In this section we study the local stability character and the boundedness nature of the solutions of Eq.(1) where the constants b and c are positive real numbers.

Note that the equilibrium points of Eq.(1) are given by the relation

$$\bar{x} = \frac{\bar{x}}{b + c\bar{x}^{-3}},$$

which gives

$$\bar{x} = 0 \quad or \quad \bar{x} = \sqrt[3]{\frac{1-b}{c}}.$$

Note that if $b < 1,$ then Eq.(1) has a unique positive equilibrium point.

Let $f : (0,\infty)^3 \to (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{w}{b + cuvw}.$$

Therefore

$$\frac{\partial f(u,v,w)}{\partial u} = \frac{-cvw^2}{(b+cuvw)^2},$$

$$\frac{\partial f(u,v,w)}{\partial v} = \frac{-cuw^2}{(b+cuvw)^2}$$

and

$$\frac{\partial f(u,v,w)}{\partial w} = \frac{b}{(b+cuvw)^2}.$$

Theorem 1 The following statements are true:

(1) If $b \geq 1,$ then $\bar{x} = 0$ is the only equilibrium point of Eq.(1) and it is locally stable.

(2) If $b < 1,$ then the equilibrium points $\bar{x} = 0$ and $\bar{x} = \sqrt[3]{\frac{1-b}{c}}$ of Eq.(1) are unstable.

Proof:

(1) If $b \geq 1,$ then we see from Eq.(4) that

$$\frac{\partial f(0,0,0)}{\partial u} = 0, \quad \frac{\partial f(0,0,0)}{\partial v} = 0,$$

$$\frac{\partial f(0,0,0)}{\partial w} = \frac{b}{(b+cuvw)^2}.$$

Then the linearized equation associated with Eq.(1) about $\bar{x} = 0$ is

$$y_{n+1} - \frac{1}{b} y_{n-3k-2} = 0, \quad (4)$$

and whose characteristic equation is

$$\lambda^{3k+3} - \frac{1}{b} = 0. \quad (5)$$

It follows by Theorem A that, Eq.(4) is asymptotically stable. Then the equilibrium point $\bar{x} = 0$ of Eq.(1) is locally stable.

(2) Assume that $b < 1.$

(i) At $\bar{x} = 0$ it follows again from Eq.(5) and Theorem A that $\bar{x} = 0$ is unstable.

(ii) At $\bar{x} = \sqrt[3]{\frac{1-b}{c}}$ we obtain

$$\frac{\partial f(\bar{x},\bar{x},\bar{x})}{\partial u} = -(1-b), \quad \frac{\partial f(\bar{x},\bar{x},\bar{x})}{\partial v} = -(1-b),$$

$$\frac{\partial f(\bar{x},\bar{x},\bar{x})}{\partial w} = b.$$

Then the linearized equation of Eq.(1) about $\bar{x} = \sqrt[3]{\frac{1-b}{c}}$ is

$$y_{n+1} + (1-b)y_{n-k} + (1-b)y_{n-2k-1} - by_{n-3k-2} = 0, \quad (6)$$

and whose characteristic equation is

$$\lambda^{3k+3} + (1-b)\lambda^{2k+2} + (1-b)\lambda^{k+1} - b = 0. \quad (7)$$

It is follows by Theorem A that, Eq.(6) is asymptotically stable if

$$|(1-b)| + |(1-b)| + |b| < 1,$$

or

$$2|(1-b)| + b < 1,$$

and so

$$2 - 2b + b < 1.$$

Therefore $1 < b$ which is a contradiction. Then the equilibrium point $\bar{x} = \sqrt[3]{\frac{1-b}{c}}$ of Eq.(1) is unstable.

The proof is complete.

In the following theorem we study the boundedness of the solutions of Eq.(1).

Theorem 2 Every solution of Eq.(1) is bounded.

Proof: Let $\{x_n\}_{n=-3k-2}^{\infty}$ be a solution of Eq.(1), we consider the following two cases

(1) If $b \geq 1.$ It follows from Eq.(1) that

$$x_{n+1} = \frac{x_{n-(3k+2)}}{b + cx_{n-k}x_{n-(2k+1)}x_{n-(3k+2)}}$$

$$\leq \frac{x_{n-(3k+2)}}{b} \leq x_{n-(3k+2)}.$$

Then the subsequences $\{x_{(3k+3)n-3k-2}\}_{n=0}^{\infty}$, $\{x_{(3k+3)n-3k-1}\}_{n=0}^{\infty},\ldots,\{x_{(3k+3)n-1}\}_{n=0}^{\infty}$, $\{x_{(3k+3)n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\left\{x_{-3k-2}, x_{-3k-1}, x_{-3k},\ldots, x_{-1}, x_0, \sqrt[3]{\frac{1}{c}}\right\}.$$

(2) If $b<1$. For the sake of contradiction, suppose that there exists a subsequence $\{x_{(3k+3)n-3k-2}\}_{n=0}^{\infty}$ is not bounded from above. Then we obtain from Eq.(1), for sufficiently large n, that

$$\infty = \lim_{n\to\infty} x_{(3k+3)n+1}$$

$$= \lim_{n\to\infty}\frac{x_{(3k+3)n-(3k+2)}}{b + cx_{(3k+3)n-k}x_{(3k+3)n-(2k+1)}x_{(3k+3)n-(3k+2)}}$$

$$< \lim_{n\to\infty}\frac{x_{(3k+3)n-(3k+2)}}{cx_{(3k+3)n-k}x_{(3k+3)n-(2k+1)}x_{(3k+3)n-(3k+2)}}$$

$$= \lim_{n\to\infty}\frac{1}{cx_{(3k+3)n-k}x_{(3k+3)n-(2k+1)}}. \qquad (8)$$

It follows that the limit of the right hand side of (8) is bounded which is a contradiction, and so the proof of the theorem is complete.

3. Solutions Form and Their Periodicity

In this section we give some different forms of the solutions of Eq.(1) whenever the coefficients b and c take different values.

Remark 1 The initial values $\{x_{-3k-2}, x_{-3k-1}, x_{-3k},\ldots, x_{-1}, x_0\}$ of Eq.(1) have not to be equal zero at the same time, otherwise Eq.(1) will have only the zero solution.

In the sequel we assume that all elements of the set $\{x_{-3k-2}, x_{-3k-1}, x_{-3k},\ldots, x_{-1}, x_0\}$ are positive real numbers.

Theorem 3 Assume that $b=c=1$ and let $\{x_n\}_{n=-3k-2}^{\infty}$ be a solution of Eq.(1). Then for $n=1,2,3,\ldots$

$$x_{(3k+3)n-3k-2} = x_{-3k-2}\prod_{i=0}^{n-1}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-3k-1} = x_{-3k-1}\prod_{i=0}^{n-1}\left(\frac{1+3ix_{-k+1}x_{-2k}x_{-3k-1}}{1+(3i+1)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-3k} = x_{-3k}\prod_{i=0}^{n-1}\left(\frac{1+3ix_{-k+2}x_{-2k+1}x_{-3k}}{1+(3i+1)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

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$$x_{(3k+3)n-2k-2} = x_{-2k-2}\prod_{i=0}^{n-1}\left(\frac{1+3ix_0x_{-k-1}x_{-2k-2}}{1+(3i+1)x_0x_{-k-1}x_{-2k-2}}\right),$$

$$x_{(3k+3)n-2k-1} = x_{-2k-1}\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-2k} = x_{-2k}\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)x_{-k+1}x_{-2k}x_{-3k-1}}{1+(3i+2)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-2k+1} = x_{-2k+1}\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)x_{-k+2}x_{-2k+1}x_{-3k}}{1+(3i+2)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

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$$x_{(3k+3)n-k-1} = x_{-k-1}\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)x_0x_{-k-1}x_{-2k-2}}{1+(3i+2)x_0x_{-k-1}x_{-2k-2}}\right),$$

$$x_{(3k+3)n-k} = x_{-k}\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+3)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-k+1} = x_{-k+1}\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)x_{-k+1}x_{-2k}x_{-3k-1}}{1+(3i+3)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-k+2} = x_{-k+2}\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)x_{-k+2}x_{-2k+1}x_{-3k}}{1+(3i+3)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

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$$x_{(3k+3)n-2} = x_{-2}\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)x_{-2}x_{-k-3}x_{-2k-4}}{1+(3i+3)x_{-2}x_{-k-3}x_{-2k-4}}\right),$$

$$x_{(3k+3)n-1} = x_{-1}\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)x_{-1}x_{-k-2}x_{-2k-3}}{1+(3i+3)x_{-1}x_{-k-2}x_{-2k-3}}\right),$$

$$x_{(3k+3)n} = x_0\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)x_0x_{-k-1}x_{-2k-2}}{1+(3i+3)x_0x_{-k-1}x_{-2k-2}}\right).$$

Proof: The proof will be achieved by Mathematical Induction. It is easy to see that the result holds for $n=1$. Now suppose that $n>1$ and our assumption holds for $n-1$; that is

$$x_{(3k+3)n-6k-5}=x_{-3k-2}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-6k-4}=x_{-3k-1}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{-k+1}x_{-2k}x_{-3k-1}}{1+(3i+1)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$\vdots$$

$$x_{(3k+3)n-5k-5}=x_{-2k-2}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{0}x_{-k-1}x_{-2k-2}}{1+(3i+1)x_{0}x_{-k-1}x_{-2k-2}}\right),$$

$$x_{(3k+3)n-5k-4}=x_{-2k-1}\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-5k-3}=x_{-2k}\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)x_{-k+1}x_{-2k}x_{-3k-1}}{1+(3i+2)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-5k-2}=x_{-2k+1}\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)x_{-k+2}x_{-2k+1}x_{-3k}}{1+(3i+2)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

$$\vdots$$

$$x_{(3k+3)n-4k-4}=x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)x_{0}x_{-k-1}x_{-2k-2}}{1+(3i+2)x_{0}x_{-k-1}x_{-2k-2}}\right),$$

$$x_{(3k+3)n-4k-3}=x_{-k}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+3)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-4k-2}=x_{-k+1}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{-k+1}x_{-2k}x_{-3k-1}}{1+(3i+3)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-4k-1}=x_{-k+2}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{-k+2}x_{-2k+1}x_{-3k}}{1+(3i+3)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

$$\vdots$$

$$x_{(3k+3)n-3k-5}=x_{-2}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{-2}x_{-k-3}x_{-2k-4}}{1+(3i+3)x_{-2}x_{-k-3}x_{-2k-4}}\right),$$

$$x_{(3k+3)n-3k-4}=x_{-1}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{-1}x_{-k-2}x_{-2k-3}}{1+(3i+3)x_{-1}x_{-k-2}x_{-2k-3}}\right),$$

$$x_{(3k+3)n-3k-3}=x_{0}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{0}x_{-k-1}x_{-2k-2}}{1+(3i+3)x_{0}x_{-k-1}x_{-2k-2}}\right).$$

Now, it follows from Eq.(1) that

$$x_{(3k+3)n-3k-2}=\frac{x_{(3k+3)n-(6k+5)}}{1+x_{(3k+3)n-(4k+3)}x_{(3k+3)n-(5k+4)}x_{(3k+3)n-(6k+5)}}$$

$$=\frac{x_{-3k-2}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right)}{1+\left(\begin{array}{c}x_{-k}x_{-2k-1}x_{-3k-2}\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+3)x_{-k}x_{-2k-1}x_{-3k-2}}\right)\\ \left(\frac{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}\right)\\ \left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right)\end{array}\right)}$$

$$=\frac{x_{-3k-2}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right)}{1+x_{-k}x_{-2k-1}x_{-3k-2}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+3)x_{-k}x_{-2k-1}x_{-3k-2}}\right)}$$

$$=\frac{x_{-3k-2}\prod_{i=0}^{n-2}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right)}{1+\left(\frac{x_{-k}x_{-2k-1}x_{-3k-2}}{1+(3n-3)x_{-k}x_{-2k-1}x_{-3k-2}}\right)}.$$

Hence, we have

$$x_{(3k+3)n-3k-2}=x_{-3k-2}\prod_{i=0}^{n-1}\left(\frac{1+3ix_{-k}x_{-2k-1}x_{-3k-2}}{1+(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 4 Assume that $b=-1$ and $c=1$ and $\{x_n\}_{n=-3k-2}^{\infty}$ be a solution of Eq.(1). Then every solution of Eq.(1) is periodic with period $(6k+6)$ and have the following form:

$$x_{(6k+6)n-3k-2}=x_{-3k-2},\quad x_{(6k+6)n-3k-1}=x_{-3k-1},$$

$$x_{(6k+6)n-3k}=x_{-3k},\ldots,$$

$$x_{(6k+6)n-1}=x_{-1},\quad x_{(6k+6)n}=x_{0},$$

$$x_{(6k+6)n+1}=\frac{x_{-3k-2}}{(-1+x_{-k}x_{-2k-1}x_{-3k-2})},$$

$$x_{(6k+6)n+2}=\frac{x_{-3k-1}}{(-1+x_{-k+1}x_{-2k}x_{-3k-1})},\ldots,$$

$$x_{(6k+6)n+k}=\frac{x_{-2k-3}}{(-1+x_{-1}x_{-k-2}x_{-2k-3})},$$

$$x_{(6k+6)n+k+1}=\frac{x_{-2k-2}}{\left(-1+x_0x_{-k-1}x_{-2k-2}\right)},$$
$$x_{(6k+6)n+k+2}=x_{-2k-1}\left(-1+x_{-k}x_{-2k-1}x_{-3k-2}\right),$$

$$x_{(6k+6)n+k+3}=x_{-2k}\left(-1+x_{-k+1}x_{-2k}x_{-3k-1}\right),...,$$
$$x_{(6k+6)n+2k+1}=x_{-k-2}\left(-1+x_{-1}x_{-k-2}x_{-2k-2}\right),$$
$$x_{(6k+6)n+2k+2}=x_{-k-1}\left(-1+x_0x_{-k-1}x_{-2k-2}\right),$$
$$x_{(6k+6)n+2k+3}=\frac{x_{-k}}{\left(-1+x_{-k}x_{-2k-1}x_{-3k-2}\right)},$$
$$x_{(6k+6)n+2k+4}=\frac{x_{-k+1}}{\left(-1+x_{-k+1}x_{-2k}x_{-3k-1}\right)},...,$$
$$x_{(6k+6)n+3k+1}=\frac{x_{-2}}{\left(-1+x_{-2}x_{-k-3}x_{-2k-4}\right)},$$
$$x_{(6k+6)n+3k+2}=\frac{x_{-1}}{\left(-1+x_{-1}x_{-k-2}x_{-2k-3}\right)},$$
$$x_{(6k+6)n+3k+3}=\frac{x_0}{\left(-1+x_0x_{-k-1}x_{-2k-2}\right)}.$$

Theorem 5 Eq.(1), with $b=-1$ and $c=1$, has a periodic solution of period $(3k+3)$ iff $x_{-k+i}x_{-2k-1+i}x_{-3k-2+i}=2\ (for\ i=0,1,2,...,k)$ and will be in the following form:

$$\{x_{-3k-2},x_{-3k-1},x_{-3k},...,x_{-1},x_0,$$
$$x_{-3k-2},x_{-3k-1},...,x_{-1},x_0,...\}.$$

Proof: First suppose that there exists a prime period $(3k+3)$ solution of Eq.(1) of the form

$$x_{-3k-2},x_{-3k-1},x_{-3k},...,x_{-1},x_0,$$
$$x_{-3k-2},x_{-3k-1},x_{-3k},...,x_{-1},x_0,....$$

Then we see from the form of solution of Eq.(1) that

$$x_{(6k+6)n-3k-2}=x_{-3k-2},x_{(6k+6)n-3k-1}=x_{-3k-1},...,$$
$$x_{(6k+6)n-1}=x_{-1},\ x_{(6k+6)n}=x_0,$$
$$x_{(6k+6)n+1}=\frac{x_{-3k-2}}{\left(-1+x_{-k}x_{-2k-1}x_{-3k-2}\right)}=x_{-3k-2},$$
$$x_{(6k+6)n+2}=\frac{x_{-3k-1}}{\left(-1+x_{-k+1}x_{-2k}x_{-3k-1}\right)}=x_{-3k-1},...,$$
$$x_{(6k+6)n+k+1}=\frac{x_{-2k-2}}{\left(-1+x_0x_{-k-1}x_{-2k-2}\right)}=x_{-2k-2},$$
$$x_{(6k+6)n+k+2}=x_{-2k-1}\left(-1+x_{-k}x_{-2k-1}x_{-3k-2}\right)=x_{-2k-1},$$
$$x_{(6k+6)n+k+3}=x_{-2k}\left(-1+x_{-k+1}x_{-2k}x_{-3k-1}\right)=x_{-2k},...,$$
$$x_{(6k+6)n+2k+2}=x_{-k-1}\left(-1+x_0x_{-k-1}x_{-2k-2}\right)=x_{-k-1},$$
$$x_{(6k+6)n+2k+3}=\frac{x_{-k}}{\left(-1+x_{-k}x_{-2k-1}x_{-3k-2}\right)}=x_{-k},...,$$
$$x_{(6k+6)n+3k+2}=\frac{x_{-1}}{\left(-1+x_{-1}x_{-k-2}x_{-2k-3}\right)}=x_{-1},$$
$$x_{(6k+6)n+3k+3}=\frac{x_0}{\left(-1+x_0x_{-k-1}x_{-2k-2}\right)}=x_0.$$

or

$$\left(-1+x_{-k}x_{-2k-1}x_{-3k-2}\right)=1,$$
$$\left(-1+x_{-k+1}x_{-2k}x_{-3k-1}\right)=1,...,$$
$$\left(-1+x_{-1}x_{-k-2}x_{-2k-3}\right)=1,$$
$$\left(-1+x_0x_{-k-1}x_{-2k-2}\right)=1.$$

Then

$$x_{-k}x_{-2k-1}x_{-3k-2}=x_{-k+1}x_{-2k}x_{-3k-1}=...$$
$$=x_{-1}x_{-k-2}x_{-2k-3}=x_0x_{-k-1}x_{-2k-2}=2.$$

Second suppose that

$$x_{-k+i}x_{-2k-1+i}x_{-3k-2+i}=2\quad (for\ i=0,1,2,...,k).$$
$$i.e.$$
$$x_{-k}x_{-2k-1}x_{-3k-2}=x_{-k+1}x_{-2k}x_{-3k-1}=...$$
$$=x_{-1}x_{-k-2}x_{-2k-3}=x_0x_{-k-1}x_{-2k-2}=2.$$

Then we see from Eq.(1) that

$$x_{(6k+6)n-3k-2}=x_{-3k-2},\quad x_{(6k+6)n-3k-1}=x_{-3k-1},...,x_{(6k+6)n-1}=x_{-1},$$

$$x_{(6k+6)n}=x_0,\ x_{(6k+6)n+1}=x_{-3k-2},...,x_{(6k+6)n+k}=x_{-2k-3},$$
$$x_{(6k+6)n+k+1}=x_{-2k-2},\ x_{(6k+6)n+k+3}=x_{-2k},...,x_{(6k+6)n+2k+1}=x_{-k-2},$$
$$x_{(6k+6)n+2k+2}=x_{-k-1},\qquad x_{(6k+6)n+2k+3}=x_{-k},$$
$$x_{(6k+6)n+2k+4}=x_{-k+1},...,x_{(6k+6)n+3k+2}=x_{-1},$$
$$x_{(6k+6)n+3k+3}=x_0.$$

Thus we have a period $(3k+3)$ solution and the proof is complete.

Theorem 6 Let $\{x_n\}_{n=-3k-2}^{\infty}$ be a solution of Eq.(1) and consider $b=1$ and $c=-1$. Then for $n=1,2,3,...$

$$x_{(3k+3)n-3k-2}=x_{-3k-2}\prod_{i=0}^{n-1}\left(\frac{1-3ix_{-k}x_{-2k-1}x_{-3k-2}}{1-(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$
$$x_{(3k+3)n-3k-1}=x_{-3k-1}\prod_{i=0}^{n-1}\left(\frac{1-3ix_{-k+1}x_{-2k}x_{-3k-1}}{1-(3i+1)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-3k}=x_{-3k}\prod_{i=0}^{n-1}\left(\frac{1-3ix_{-k+2}x_{-2k+1}x_{-3k}}{1-(3i+1)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

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$$x_{(3k+3)n-2k-2}=x_{-2k-2}\prod_{i=0}^{n-1}\left(\frac{1-3ix_{0}x_{-k-1}x_{-2k-2}}{1-(3i+1)x_{0}x_{-k-1}x_{-2k-2}}\right),$$

$$x_{(3k+3)n-2k-1}=x_{-2k-1}\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)x_{-k}x_{-2k-1}x_{-3k-2}}{1-(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-2k}=x_{-2k}\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)x_{-k+1}x_{-2k}x_{-3k-1}}{1-(3i+2)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-2k+1}=x_{-2k+1}\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)x_{-k+2}x_{-2k+1}x_{-3k}}{1-(3i+2)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

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$$x_{(3k+3)n-k-1}=x_{-k-1}\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)x_{0}x_{-k-1}x_{-2k-2}}{1-(3i+2)x_{0}x_{-k-1}x_{-2k-2}}\right),$$

$$x_{(3k+3)n-k}=x_{-k}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-k}x_{-2k-1}x_{-3k-2}}{1-(3i+3)x_{-k}x_{-2k-1}x_{-3k-2}}\right),$$

$$x_{(3k+3)n-k+1}=x_{-k+1}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-k+1}x_{-2k}x_{-3k-1}}{1-(3i+3)x_{-k+1}x_{-2k}x_{-3k-1}}\right),$$

$$x_{(3k+3)n-k+2}=x_{-k+2}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-k+2}x_{-2k+1}x_{-3k}}{1-(3i+3)x_{-k+2}x_{-2k+1}x_{-3k}}\right),$$

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$$x_{(3k+3)n-2}=x_{-2}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-2}x_{-k-3}x_{-2k-4}}{1-(3i+3)x_{-2}x_{-k-3}x_{-2k-4}}\right),$$

$$x_{(3k+3)n-1}=x_{-1}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-1}x_{-k-2}x_{-2k-3}}{1-(3i+3)x_{-1}x_{-k-2}x_{-2k-3}}\right),$$

$$x_{(3k+3)n}=x_{0}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{0}x_{-k-1}x_{-2k-2}}{1-(3i+3)x_{0}x_{-k-1}x_{-2k-2}}\right).$$

Theorem 7 Let $\{x_n\}_{n=-3k-2}^{\infty}$ be a solution of Eq.(1) with $b=c=-1$. Then every solution of Eq.(1) is periodic with period $(6k+6)$ and for $n=1,2,3,...$

$$x_{(6k+6)n-3k-2}=x_{-3k-2},\quad x_{(6k+6)n-3k-1}=x_{-3k-1},$$

$$x_{(6k+6)n-3k}=x_{-3k},\ldots,$$

$$x_{(6k+6)n-1}=x_{-1},\quad x_{(6k+6)n}=x_{0},$$

$$x_{(6k+6)n+1}=\frac{x_{-3k-2}}{(-1-x_{-k}x_{-2k-1}x_{-3k-2})},$$

$$x_{(6k+6)n+2}=\frac{x_{-3k-1}}{(-1-x_{-k+1}x_{-2k}x_{-3k-1})},\ldots,$$

$$x_{(6k+6)n+k}=\frac{x_{-2k-3}}{(-1-x_{-1}x_{-k-2}x_{-2k-3})},$$

$$x_{(6k+6)n+k+1}=\frac{x_{-2k-2}}{(-1-x_{0}x_{-k-1}x_{-2k-2})},$$

$$x_{(6k+6)n+k+2}=x_{-2k-1}(-1-x_{-k}x_{-2k-1}x_{-3k-2}),$$

$$x_{(6k+6)n+k+3}=x_{-2k}(-1-x_{-k+1}x_{-2k}x_{-3k-1}),\ldots,$$

$$x_{(6k+6)n+2k+1}=x_{-k-2}(-1-x_{-1}x_{-k-2}x_{-2k-2}),$$

$$x_{(6k+6)n+2k+2}=x_{-k-1}(-1-x_{0}x_{-k-1}x_{-2k-2}),$$

$$x_{(6k+6)n+2k+3}=\frac{x_{-k}}{(-1-x_{-k}x_{-2k-1}x_{-3k-2})},$$

$$x_{(6k+6)n+2k+4}=\frac{x_{-k+1}}{(-1-x_{-k+1}x_{-2k}x_{-3k-1})},\ldots,$$

$$x_{(6k+6)n+3k+1}=\frac{x_{-2}}{(-1-x_{-2}x_{-k-3}x_{-2k-4})},$$

$$x_{(6k+6)n+3k+2}=\frac{x_{-1}}{(-1-x_{-1}x_{-k-2}x_{-2k-3})},$$

$$x_{(6k+6)n+3k+3}=\frac{x_{0}}{(-1-x_{0}x_{-k-1}x_{-2k-2})}.$$

Theorem 8 Assume that $b=c=-1$. Then Eq.(1) has a periodic solution of period $(3k+3)$ iff $x_{-k+i}x_{-2k-1+i}x_{-3k-2+i}=-2$ ($for\ i=0,1,2,...,k$) and will be take the form

$$\{x_{-3k-2}, x_{-3k-1}, x_{-3k}, \ldots, x_{-1}, x_0,$$
$$x_{-3k-2}, x_{-3k-1}, x_{-3k}, \ldots, x_{-1}, x_0, \ldots\}.$$

The following theorem deals with the periodic solutions of the general form of Eq.(1).

Theorem 9 Eq.(1) has positive prime period $(3k+3)$ solutions if and only if

$$(b+cA_i-1)=0, \qquad (9)$$

where $A_i = x_{-k+i}x_{-2k-1+i}x_{-3k-2+i}$ $(\, for\ i=0,1,2,\ldots,k)$ and $A_{k+1+i}=A_i$.

Proof: First suppose that there exists a prime period $(3k+3)$ solution of Eq.(1) of the form

$$\ldots, x_{-3k-2}, x_{-3k-1}, x_{-3k}, \ldots, x_{-1}, x_0,$$
$$x_{-3k-2}, x_{-3k-1}, x_{-3k}, \ldots, x_{-1}, x_0, \ldots.$$

That is $x_{N+1}=x_{N-3k-2}$ $for\ N\geq 0$. We will prove that (9) holds.
Now we see from Eq.(1) that

$$x_{-3k-2}=x_1=\frac{x_{-3k-2}}{b+cA_0}, \quad x_{-3k-1}=x_2=\frac{x_{-3k-1}}{b+cA_1},$$

$$x_{-3k}=x_3=\frac{x_{-3k}}{b+cA_2}, \ldots,$$

$$x_{-2k-2}=x_{k+1}=\frac{x_{-2k-2}}{b+cA_k},$$

$$x_{-2k-1}=x_{k+2}=\frac{x_{-2k-1}}{b+cA_{k+1}}=\frac{x_{-2k-1}}{b+cA_0}, \ldots,$$

$$x_{-k-2}=x_{2k+1}=\frac{x_{-k-2}}{b+cA_{2k}}=\frac{x_{-k-2}}{b+cA_{k-1}},$$

$$x_{-k-1}=x_{2k+2}=\frac{x_{-k-1}}{b+cA_{2k+1}}=\frac{x_{-2k-1}}{b+cA_k}, \ldots,$$

$$x_{-2}=x_{3k+1}=\frac{x_{-2}}{b+cA_{3k}}=\frac{x_{-2}}{b+cA_{k-2}},$$

$$x_{-1}=x_{3k+2}=\frac{x_{-1}}{b+cA_{3k+1}}=\frac{x_{-1}}{b+cA_{k-1}},$$

$$x_0=x_{3k+3}=\frac{x_0}{b+cA_{3k+2}}=\frac{x_0}{b+cA_k}.$$

Then it is easy to see that

$$x_{-3k-2}(b+cA_0)=x_{-3k-2} \Rightarrow x_{-3k-2}(b+cA_0-1)=0,$$

$$x_{-3k-1}(b+cA_1)=x_{-3k-1} \Rightarrow x_{-3k-1}(b+cA_1-1)=0,$$

$$x_{-3k}(b+cA_2)=x_{-3k} \Rightarrow x_{-3k}(b+cA_2-1)=0,$$
$$x_{-1}(b+cA_{k-1})=x_{-1} \Rightarrow x_{-1}(b+cA_{k-1}-1)=0,$$

$$x_0(b+cA_k)=x_0 \Rightarrow x_0(b+cA_k-1)=0.$$

Since $x_j \neq 0$ for all $-(3k+2)\leq j\leq 0$, then Condition (9) is satisfied.

Second suppose that (9) is true. We will show that Eq.(1) has a prime period $(3k+3)$ solution.

It follows from Eq.(1) and Eq.(9) that

$$x_1=\frac{x_{-3k-2}}{b+cA_0}=x_{-3k-2}, \quad x_2=\frac{x_{-3k-1}}{b+cA_1}=x_{-3k-1},$$

$$x_3=\frac{x_{-3k}}{b+cA_2}=x_{-3k}, \ldots,$$

$$x_{k+1}=\frac{x_{-2k-2}}{b+cA_k}=x_{-2k-2},$$

$$x_{k+2}=\frac{x_{-2k-1}}{b+cA_{k+1}}=\frac{x_{-2k-1}}{b+cA_0}=x_{-2k-1}, \ldots,$$

$$x_{2k+1}=\frac{x_{-k-2}}{b+cA_{2k}}=\frac{x_{-k-2}}{b+cA_{k-1}}=x_{-k-2},$$

$$x_{2k+2}=\frac{x_{-k-1}}{b+cA_{2k+1}}=\frac{x_{-2k-1}}{b+cA_k}=x_{-k-1}, \ldots,$$

$$x_{3k+1}=\frac{x_{-2}}{b+cA_{3k}}=\frac{x_{-2}}{b+cA_{k-2}}=x_{-2},$$

$$x_{3k+2}=\frac{x_{-1}}{b+cA_{3k+1}}=\frac{x_{-1}}{b+cA_{k-1}}=x_{-1},$$

$$x_{3k+3}=\frac{x_0}{b+cA_{3k+2}}=\frac{x_0}{b+cA_k}=x_0,$$

which completes the proof.

4. Global Stability of the Solutions

Here we study the convergent of the solutions of Eq.(1).

Theorem 10 If $b\geq 1$, then every solution of Eq.(1) converges to the equilibrium point $\bar{x}=0$.

Proof: It was shown in Theorem 1 that $\bar{x}=0$ is local stable and then it suffices to show that $\bar{x}=0$ is global attractor of the solutions of Eq.(1).

We claim that each one of the subsequences $\{x_{(3k+3)n-3k-2}\}_{n=0}^{\infty}$, $\{x_{(3k+3)n-3k-1}\}_{n=0}^{\infty}, \ldots,$ $\{x_{(3k+3)n-1}\}_{n=0}^{\infty}$, $\{x_{(3k+3)n}\}_{n=0}^{\infty}$ has limit equal to zero. For the sake of contradiction, suppose that there exists a subsequence $\{x_{(3k+3)n-3k-2}\}_{n=0}^{\infty}$ with limit doesn't zero. Now we see from Eq.(1) that

$$bx_{(3k+3)n+1}+cx_{(3k+3)n+1}x_{(3k+3)n-k}x_{(3k+3)n-(2k+1)}x_{(3k+3)n-(3k+2)} = x_{(3k+3)n-(3k+2)},$$

or

$$x_{(3k+3)n-(3k+2)} = \frac{bx_{(3k+3)n+1}}{1-cx_{(3k+3)n+1}x_{(3k+3)n-k}x_{(3k+3)n-(2k+1)}}.$$

Now it follows from the boundedness of the solution that

$$\lim_{n\to\infty} x_{(3k+3)n-(3k+2)} = \lim_{n\to\infty}\frac{bx_{(3k+3)n+1}}{1-cx_{(3k+3)n+1}x_{(3k+3)n-k}x_{(3k+3)n-(2k+1)}} < \frac{bM}{1-cM^3} < 0,$$

where $M \geq \sqrt[3]{\frac{1}{c}}$ which is a contradiction and this completes the proof of the theorem.

5. Numerical Examples

For confirming the results of this paper, we present some numerical examples which show the behavior of solutions of Eq.(1). See below Figure 1 ($b=2, c=4,$ $k=1,$ $x_{-5}=10, x_{-4}=7,$ $x_{-3}=3,$ $x_{-2}=12,$ $x_{-1}=6,$ $x_0=4.$), Figure 2 ($b=0.4,$ $c=0.8,$ $k=1,$ $x_{-5}=15,$ $x_{-4}=7,$ $x_{-3}=0.5,$ $x_{-2}=4,$ $x_{-1}=9,$ $x_0=4$.) and Figure 3 ($b=-3,$ $c=2,$ $k=2,$ $x_{-8}=6,$ $x_{-7}=-7,$ $x_{-6}=0.5,$ $x_{-5}=-4,$ $x_{-4}=1/7,$ $x_{-3}=4,$ $x_{-2}=-1/12,$ $x_{-1}=-2,$ $x_0=1$.).

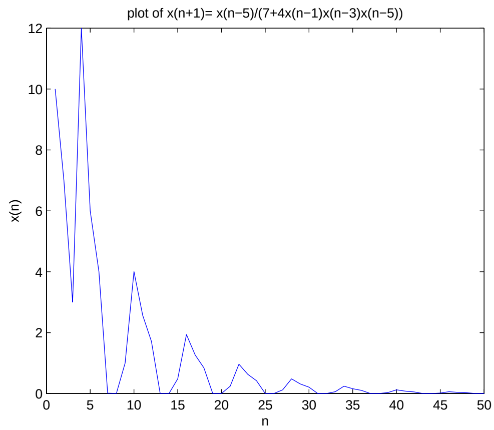

Figure 1

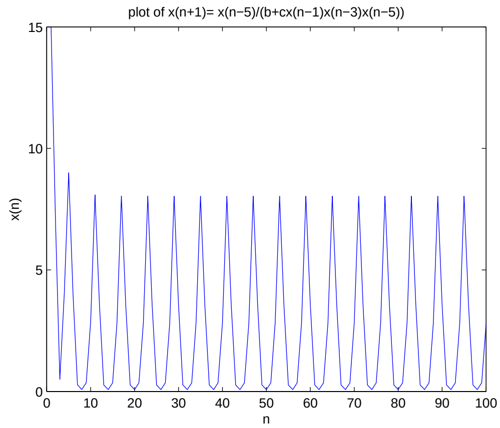

Figure 2

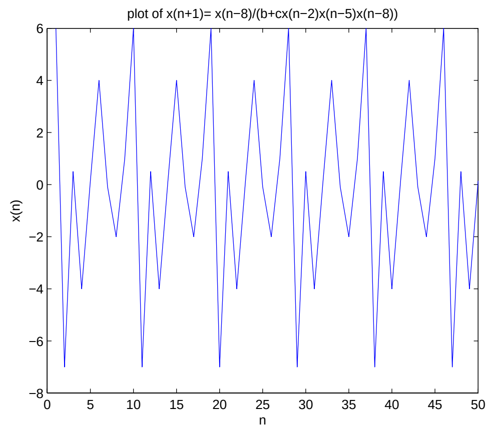

Figure 3

The following figures show the behavior of the solutions of Eq.(1) (when $b=c=1$) with a fixed order and some numerical values of the initial values:

Figure 4: $k=3$, $x_{-11}=8$, $x_{-10}=11$, $x_{-9}=4$, $x_{-8}=7$, $x_{-7}=13$, $x_{-6}=6$, $x_{-5}=3$, $x_{-4}=5$, $x_{-3}=1$, $x_{-2}=9$, $x_{-1}=2$, $x_0=10$. Figure 5: $k=3$, $x_{-11}=0.5$, $x_{-10}=1.3$, $x_{-9}=0.2$, $x_{-8}=0.7$, $x_{-7}=1.1$, $x_{-6}=0.6$, $x_{-5}=0.9$, $x_{-4}=0.5$, $x_{-3}=1$, $x_{-2}=0.1$, $x_{-1}=1.4$, $x_0=0.2$.

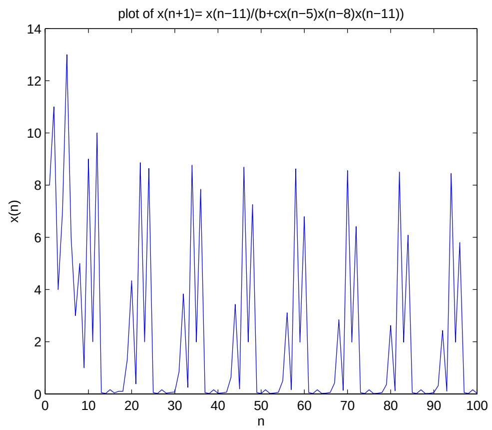

Figure 4

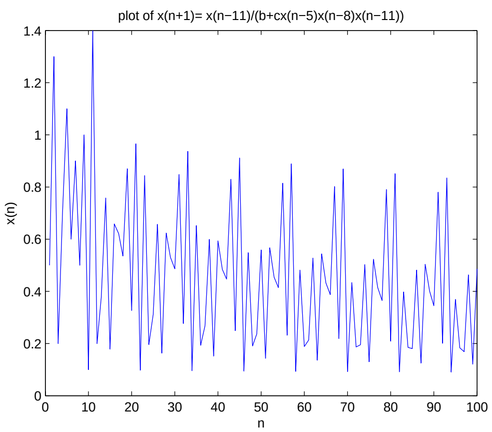

Figure 5

In the following we give some numerical examples to confirm the obtained results for Eq.(1) (when $b=-1, c=1$). See below Figures 6 ($k=2$, $x_{-8}=0.2$, $x_{-7}=0.5$, $x_{-6}=1.8$, $x_{-5}=0.9$, $x_{-4}=-2.1$, $x_{-3}=1.3$, $x_{-2}=2.5$, $x_{-1}=1.7$, $x_0=2.7$.) and Figure 7 ($k=2$, $x_{-8}=2$, $x_{-7}=0.5$, $x_{-6}=6$, $x_{-5}=1/9$, $x_{-4}=-5$, $x_{-3}=1/9$, $x_{-2}=9$, $x_{-1}=-4/5$, $x_0=6$.).

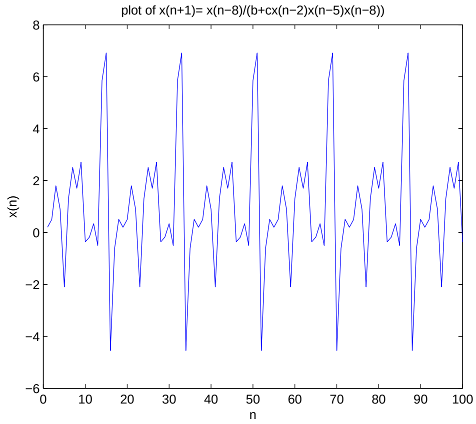

Figure 6

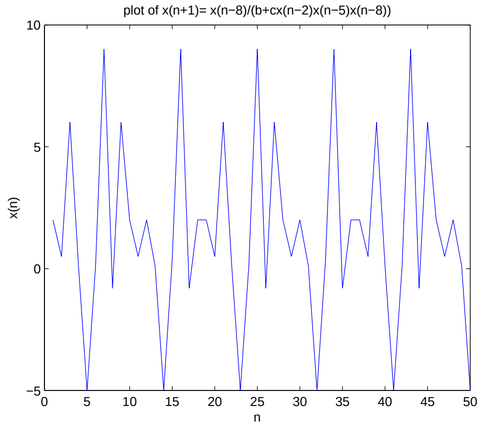

Figure 7

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