On the simplicial cohomology theory of algebra

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Abstract: In this article we study the Hochschild (simplicial), cohomology of Hochschild complex of A_∞ –algebra with some homotopical properties. We study also The relation between the Hochschild cohomology of commutative A_∞ -algebra and the set of twisted cochain D(A,A) of this complex. We prove that the vanishing of Hochschild cohomology of special degree leads to vanish of D(A,A) . In the third part we get an extension of special case of A_∞ –algebra. 2000 Mathematics Subject Classification: 55N35,16E4.

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1. Introduction

The concepts of A_∞ -modules, A_∞ -algebra and its related with simplicial (Hochschild) (co)homology has been studied in [2,4,6,7]. The relationship between a set of all A_∞ -algebra structures on fixed differential graded algebra and the Hochschild cohomology of that algebra has been studied in [7]. The Hochschild cohomology complex for A_∞ -modules over A_∞ - algebras has been studied in [5]. The triviality of the Hochschild cohomology of dimension $(n,\ 2-n)$ A_∞ -algebra is proved in [4]. In this article we are concerned by (Hochschild cohomology) of differential A_∞ -algebra. For a given cochain Hochschild (simplicial) complex for differential A_∞ – algebra with finite number of non trivial high multiplication π_i and differential A_∞ –algebra A, we show that the Hochschild cohomology is trivial. We generalized the triviality of the Hochschild cohomology in [4].

2 Hochschild complex for differential A_∞ -algebra

In this part we recall the requisites definitions and results relating to concepts of D_∞ differential module and D_∞ - differential algebra. The main references are [3], [4], [5],[7] and [9].
Note that all modules are defined on Z_2.
Definition 2.1:

A differential module (X,d) is a module $X=\{x_n\}$, $n\in\mathbb{Z}$ equipped with a morphism $d: X\to X$, called the differential of the module X, with degree (−1) such that $d^2=0$.
A mapping of differential modules $f:(X,d)\to(Y,d)$ is a mapping of modules $f: X\to Y$ such that $df=fd$.
A differential homotopy between mappings $f,g:(X,d)\to(Y,d)$ of differential modules is a mapping of module $h: X\to Y$ module X , where the space $Z_n(X)=Ker\{d_n: X_n\to X_{n-1}\}$ is called a space n-dimensional cycles and spaces $B_n(X)=\operatorname{Im}\{d_{n+1}: X_{n+1}\to X_n\}$ are called spaces of n-dimensional boundaries. It is clear that $B_n(X)\subset Z_n(X)$ such that $dh+hd=f-g$.
It is easy to see that the homotopy relation is an equivalence relation.
The modules X and Y are called homotopy equivalent (denoted by $X\approx Y$), if there is a chain map $f: X\to Y$, $g: Y\to Z$ such that $g\circ f\approx Id_X$, $f\circ g\ \approx Id_Y$.

The module is called contractible, if it is homotopically equivalent to the zero. The factor space $H_n(X)=Z_n(X)/B_n(X)$ is called the homology of X.

For a module X, denote by $\overline{X}$ the dual module of X, $\overline{X}=\{\overline{X}_n\}$, for which X_{-n} - conjugate to X_{-n}. The differentials $d_n : X_n \to X_{n-1}$ induce the differentials $\overline{d}_n : \overline{X}_{-n+1} \to \overline{X}_{-n}$. The homology of the dual complex $\overline{X}$ is called cohomology of X and denoted by $H^*(X)=\{H^n(X), H^n(X)=H_{-n}(\overline{X})$.

Definition 2.2:

A D_∞- differential module (X,d^i) is a an arbitrary Banach module X with a family of homeomorphisms $\{d^i : X \to X, i \ge 0\}$ such that the following relations holds for each integer $k \ge 0_k$, $\sum_{i+j=k} d^i d^j = 0$.

If $i=0$, $d^0 d^0 = 0$ and (X,d^0) is an ordinary differential module, if *i=1* we have $d^1 d^0 + d^0 d^1 = 0$, that is the mappings d^0 and d^1 are anticommuting maps. This means that the composition $d^1 d^1 : X \to X$ is an endomorphism of the differential module (X,d^0). For k = 2, we obtain $d^2 d^0 + d^0 d^2 = 0 - d^1 d^1$.This means that the mapping $d^2 : X \to X$ is a differential homotopy between the zero map and the mapp $d^1 d^1 : (X,d^0) \to (X,d^0)$ of differential modules. Therefore, the mapping $d^1 : X \to X$ is a differential within a homotopy.

Definition 2.3:

A differential algebra (A,d,π) is a differential module (A,d) over algebra with the multiplication $\pi : A \otimes A \to A$ such that the associate law $(\pi \otimes 1)\pi = (1 \otimes \pi)\pi$ holds.

Definition 2.4 :

Let A be algebra. The triple (A,d,π_i) is called A_∞–algebra , where (A,d) is graded module over algebra such that:

$$\sum_{i=0}^{n} (-1)^\varepsilon \pi_i (1 \otimes \ldots \otimes \pi_{n-i} \otimes \ldots \otimes 1) = 0, \qquad (1)$$
$$\varepsilon = nk + ik + n + k$$

The morphism between A_∞-algebras A, A` is a family of homeomorphism $f=\{f_i : A^{\otimes i} \to A'\}$ such that $f_i((A^i))_q \in A'_{q+i-1} \to A'\}$ and

$$\sum_j f_{i-j+1}(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes \ldots \otimes 1) = \sum \pi'_\ell (f_{k_1} \otimes f_{k_2} \otimes \ldots \otimes f_{k_\ell}) \qquad (2)$$

The summation in (1) and (2) are given in all possible place of m_j and the right hand side of (2) we can put :

$i = k_1 + k_2 + + \ldots + k_\ell$ .The forms (1) and (2) are called stasheff relation for A_∞-algebra [7].

Definition 2.5 :

A differential algebra A_∞-algebra is D_∞- module A tog- ether with a set operations $\pi_n : A^{\otimes n+2} \to A$, $n \ge 0$, with the following identity:

$$d(\pi_{n+1}) = \sum_{i=0}^{n} (-1)^\varepsilon \pi_i (1 \otimes \ldots \otimes \pi_{n-i} \otimes \ldots \otimes 1),$$
$$\varepsilon = nk + ik + n + k$$

For example if *n=0*, then $d(\pi_1) = \pi_0(-\pi_0 \otimes 1 + 1 \otimes \pi_0)$, this is associated homotopy relation.

If *n=1* then

$$d(\pi_2) = \pi_0(\pi_1 \otimes 1 + 1 \otimes \pi_1) + \pi_1(\pi_0 \otimes 1 \otimes 1 - 1 \otimes \pi_0 \otimes 1 + 1 \otimes 1 \otimes \pi_0)$$

this means that there is a homotopy relation between π_0 and π_1.

Definition 2.6 :

The module *A* is called differential coalgebra, if there is a specified operation $\nabla_n : A \to A^{\otimes n+2}$ of dimensions $n \ge 0$, satisfying the relations :

$$d(\nabla_{n+1}) = \sum_{i=0}^{n} (-1)^\varepsilon (1 \otimes \ldots \otimes \nabla_{n-i} \otimes \ldots \otimes 1)\nabla_i,$$
$$\varepsilon = nk + ik + ni + i + k + 1$$

from [4] the Hochschild complex $C^*(A, A)$ for algebras *A* is a A-module over Z_2 with the multiplication π:A$\otimes A \to A$ with the associate law $(\pi \otimes 1)\pi = (1 \otimes \pi)\pi$.

The cochain Hochschild complex is given by $(C^*(A, A), \delta)$ such that

$C^*(A, A) = \sum C^n(A, A), \delta)$,

$C^*(A, A) = Hom(A^n, A)$ and

$\delta : C^n(A, A) \to A^{n+1}(A, A)$.

The relation between operators δ and π is given by:

$$\delta f = \pi(1 \otimes f) + \sum f(1 \otimes \ldots \otimes \pi \otimes \ldots \otimes 1) + \pi(f \otimes 1)$$

The homology of $(C^*(A, A), \delta)$ is Hochschild cohomology and defined by $H^*(A, A)$.

Definition 2.7 :

For differential A_∞-algebra A we can define the coalgebra BA which is called B-construction over A. Consider the tensor algebra $TA = \sum_{n \ge 1} A^{\otimes n}$ such that: $\deg(a_1 \otimes \ldots \otimes a_k) = \deg(a_1) + \ldots \deg(a_k) + k$. The tensor algebra *TX* with the following differential $d : BA_i \to A_{i-1}$, such that

$$d(a_1 \otimes \ldots \otimes a_2) = \sum_{k,j} a_1 \otimes \ldots \otimes \pi_k(a_j \otimes \ldots \otimes a_{j+k)} \otimes \ldots \otimes a_n$$

is called *B*-construction over *A* and denoted by BA .

We consider the differential A_∞-algebra A with finite integer nontrivial exterior multiplication π_i, then there is A_∞-algebra such that for $, n \in Z$, $\pi_i = 0$, for $i > n$.

Consider

$Hom(BA, A)$ then

$Hom^n(BA, A) = [f : BA_i \to A_{i+n}]$.

Note that $f \in Hom^n(BA, A)$, then there is $\{f_i\}$,

$f_i : (A^i)_q \to A_{q+i+n}$. The identity map is $Id_1 = d$, $Id_k = 0$ for k>1 .

Define the differential

$\delta : Hom^n(BA, A) \to Hom^{n-1}(BA, A)$ such that

$$\delta f = \sum_i f(1 \otimes \ldots \otimes 1 \otimes \pi_i \otimes 1 \otimes \ldots \otimes) + \sum_i \pi_i(1 \otimes \ldots \otimes f \otimes \ldots \otimes 1) \quad (3)$$

The complex $Hom(BA, A)$ with differential δ (defined in relation (3)) is called the Hochschild complex for A_∞–algebra and denoted by $C_\infty(A, A)$.

Consider the following operations in Hochschild complex C(A, A) from [4]:

$f \cup g = \pi(f \otimes g)$,

$f \cup_1 g = \sum_k f(1 \otimes \ldots \otimes 1 \otimes g \otimes \ldots \otimes 1)$ where

$f \in C^m(A, A), g \in C^n(A, A)$.

We can rewrite the operations $\cup$ and $\cup_1$ on the Hochschild complex $C_\infty(A, A)$ as follows :

$$\cup : (C_\infty(A, M) \otimes C_\infty(A, A))^i \to (C_\infty(A, M))^i,$$

$$f \cup g = f(1 \otimes \ldots \otimes 1 \otimes g),$$

$$\cup_1 : (C_\infty(A, A) \otimes C_\infty(A, A))^i \to C_\infty^{i+1}(A, A), \quad (5)$$

$$f \cup_1 g = \sum f(1 \otimes \ldots \otimes 1 \otimes g \otimes \ldots \otimes 1)$$

where $f, g \in C_\infty(A, A)$.

For some $(g_1, g_2, \ldots, g_k)$ we can generalize the operation $\cup_1$ in relation (5) to be

$$\cup_i^k : (C_\infty(A, A)^{\otimes k+1})^i \to C_\infty^{i+k}(A, A),$$

$$f \cup_1^k (g_1, g_2, \ldots, g_k) = \sum f(1 \otimes \ldots \otimes 1 \otimes g_1 \otimes 1 \otimes \ldots \otimes 1 \otimes g_2 \otimes 1 \otimes \ldots \otimes 1 \otimes g_k \otimes 1 \otimes \ldots \otimes 1), \quad (6)$$

$f, g \in C_\infty(A, A)$ and the summation will be in all place of elements $g_1, g_2, \ldots, g_k$.

The relation between operators $\cup_1$, δ and π is given by:

$$\delta f = f \cup_1 \pi + \pi \cup_1 f$$

From relation (3), (4), (5), (6) we get :

$$\delta(f \cup_1^k (g_1, g_2, \ldots, g_k)) = \delta f \cup_1^k (g_1, g_2, \ldots, g_k) + \sum_{i=1}^{k} f \cup_1^k (g_1, g_2, \ldots, g_k) + \sum_{i=2}^{k} \sum_{s=1}^{k-2} f \cup_1^{k-i+1} (g_1, \ldots, \pi \cup_1^i (g_s, \ldots, g_{s+i}), \ldots, g_k) + \sum_{i=0}^{k-1} \sum_{s=1}^{k-2} \pi \cup_1^{k-i+1} (g_1, \ldots, f \cup_1^i (g_s, \ldots, g_{s+i}), \ldots, g_k). \quad (7)$$

The relation (7), when k=1, can be written in the form:

$$\delta(f \cup_1 g) = (\delta f) \cup_1 g + f \cup_1 (\delta g) + f \cup g + g \cup f,$$

if we put $\cup_1^1 = \cup$ and $\pi \cup_1^2 (f, g) = f \cup g$ in (7) and $f \cup_1^0 = f$ in (5)

3- Twisted cochain Hochschild complex for A_∞-algebra and related cohomology.

In this part we are concerned with the commutative A_∞-algebra and triviality of the Hochschild cohomology in [4]. We define a new concept of twisted cochain on Hochschild complex $C_\infty(A,A)$ for A_∞-algebra and get the theorems (2.4) and (2.5) analog to theorems of kadishfili in [4].

Firstly we recall the definition of commutative A_∞-algebra and its related cohomology, we also define the twisted cochain and its propereties on Hochschild complex from [1], [4] and [7].

Definition 3.1.

The twisted cochain is an element $a = a^{3,-1} + a^{4,-2} + \ldots + a^{i,2-i} + \ldots$ where $a^{i,2-i} \in C^{i,2-i}(A,A)$, such that $\delta a = a \cup_1 a$, since $\cup_1$ is multiplication in the Hochschild complex for algebra A. The set of twisted cochains is denoted by $TW(A,A)$.

Definition 3.2.

Two twisted cochain a and a' are equivalent $(a \sim a')$ if there exist an element $p = p^{2,-1} + p^{3,-2} + \ldots + p^{i,1-i}$, $p^{i,1-i} \in C^{i,1-i}(A,A)$ such that:

$$a - a' = \delta p + p \cup_1 a + a' \cup_1 (p \otimes p) + a' \cup_1 (p \otimes p \otimes p) + \ldots \ .$$

The set $TW(A,A)/\sim$, where $\sim$ is an equivalent relation, is denoted by $D(A,A)$.

In the following we define the A_∞-algebra commutative case and its related cohomology.

Definition: A_∞-algebra A is commutative algebra if $\sum_\sigma (-1)^\varepsilon m_n \sigma(i, n-i) = 0$, where the summation is got on the perturbation σ.

Definition : If A is commutative algebra, then it's Hochschild complex $C^*((A,A),\delta)$ is called Harresona complex.

Definition 3.3 .

The cohomology of the complex $C^*((A,A),\delta)$ is called Harresona cohomology of commutative algebra A and denoted by , then it's Hochschild complex $Harr(C^*(A,A),\delta))$ is called Harresona complex.

In the following we define a new concept of twisted cochain on Hochschild complex $C_\infty(A,A)$ for A_∞-algebra.

Definition 3.4.

Any element $g \in C_\infty^{-2}(A,A)$ is called twisted cochain if the following hold:

1. $g_i = 0, \ if \quad i < n+1$
2. $\delta g = g \cup_1 g$ (8)

The set of all twisted cochain in Hochschild complex $C_\infty(A,A)$ is denoted $TW(C_\infty(A,A))$.

Definition 3.5.

Two twisted cochains g and g' are equivalent and denoted by:

$g \sim g'$ if there is $f \in C_\infty^{-1}(A,A)$, such that:

1- $f_1 = id$,

$$2 - \delta f + \sum_{i=2} \pi \cup_1^i (f, \ldots, f) + f \cup_1 g' + \sum_{i=1} g \cup_1^i (f, \ldots, f) = 0 \qquad (9)$$

Where $\cup_1$ and $\cup_1^i$ are defined by formula (5), (6).

Suppose that $D(A,A) = TW(C_\infty(A,A)/\sim)$ where $\sim$ is an equivalent relation, then the following holds.

Theorem 3.6.

Let $g \in TW(C_\infty(A,A))$ be an arbitrary twisted cochain and $f \in C_\infty^{-1}(A,A)$, such that $f_1 = id$, $f_i = 0$ for $i > n+1$, then there exist Twisted cochain $\overline{g}$ such that:

1. $g_i = \overline{g}_i, \ i < k+1, k > n$;
2. $\overline{g}_{k+1} = g_{k+1} + (\delta f)_{k+1}$,
3. $\overline{g} \sim g$
4. Proof. We use the method of constructing element $\overline{g} \sim g$. Note that , to use the condition of the theorem 2.4 we have the relation $(\delta f)_{n+1} = \delta(f_n)$.

For every element f in definition 3.3, which make the equivalent relation $\overline{g} \sim g$, we consider it as an element satisfies condition of theorem 2.4. Define $\overline{g}_i, \ i < k+1$ from condition 1 of theorem 2.4. For elements g and $\overline{g}$, the first nontrivial elements in right hand side of relation (9) is given in (k+1)-dimension, such that $\delta f + f_1(g) + f_1(\overline{g}) = 0$, this relation is true if $f_1 = id$ (all remain $f_i = 0$, for $1 < i < k+1$).

Theorem 3.7.

If Let $H^{-2}(C_\infty(A,A))=0$, then $D(A,A)=0$.

Proof: we must prove that the arbitrary twisted cochain, given condition, is equal zero. The formula (8), for element g, in (n+1)-dimension has the form $\delta g=0$, that is g is acyclic. By considering the condition $H^{-2}(C_\infty(A,A))=0$ there exist f^1 such that $g_{n+1}=(\delta f^1)_{n+1}$ or $g-0=\delta f$. Following theorem 3.6 we can get a twisted cochain g^1 such that $g^1_{n+1}=0$ and $g\sim g^1$. Hence the formula (8) in (n+2)-dimension, for element g^1, is given by $\delta g^1=0$, that is g^1 is acyclic. since $H^{-2}(C_\infty(A,A))=0$, then there is f^2 such that $g_{n+1}=(\delta f^1)_{n+1}$ or $g^1-0=\delta f^2$ and so on. Repeating this process we get a sequence of twisted cochain such that $g^i_{n+k}=0,\ k<i+1$. The extension of this process to infinity get trivial twisted cochain with the element f with components $f_1=0$ and $f_i=0,$ for

$$i<n+1,\ f_i=f_i^{n-i},\ i>n$$

4- Extension of A∞ -algebra and cohomology of Hochschild of C∞(A, A) for A∞ -algebra

Definition (4.1) .

Let A be A∞ -algebra (A,π_i) with nontrivial finite number of the multiplication π_i i.e. $(\pi_i\neq 0,\ 0\leq i\leq n,\ ,\pi_i=0,\ i>n$.

The extension of A∞-algebra is an A∞-algebra Ā such that A and Ā coincided and the high multiplication $\overline{\pi}_i=\pi_i$, for $i<n+1$.

In [4] is proved that there is a bijection between the set of structure A∞ -algebra on fixed graded algebra, such that $\pi_1=0$, $\pi_2=\pi$, π is multiplication in algebra, denoted by $(A,\pi)(\infty)$ and the set of twisted cochains Hochschild complex factored by the equivalent relation ~.

Here we give an extension of this fact between the set of all extension of a fixed A_∞-algebra, denoted by $(A,\pi_i)(\infty)$, where π_i is the structure on a fixed A_∞-algebra A, and the set of twisted cochains Hochschild complex factored by the equivalent relation ~(D(A,A)).

The following theorem is the main result in this part.

Theorem (4.2): There is a bijection between sets $(A,\pi_i)(\infty)$ and $D(A,A)$.

Proof: For $A\in(A,\pi_i)(\infty)$ consider the stasheff relation (1) as follow:

$$\sum_{i=1,j=1}^{n}\pi_i(1\otimes\ldots\otimes 1\otimes\pi_j\otimes 1\otimes\ldots\otimes 1)+$$
$$+\sum_{i=1,j=n+1}^{n}\pi_i(1\otimes\ldots\otimes 1\otimes\pi_j\otimes 1\otimes\ldots.\otimes 1)+$$
$$+\sum_{i=n+1,j=1}^{n}\pi_i(1\otimes\ldots\otimes 1\otimes\pi_j\otimes 1\otimes\ldots\otimes 1)+$$
$$+\sum_{i,j=n+1}^{n}\pi_i(1\otimes\ldots\otimes 1\otimes\pi_j\otimes 1\otimes\ldots\otimes 1)=0 \qquad (10)$$

Clearly that the first term of (10) is equal zero, following stasheff relation for fixed algebra A. The second and third terms of (10) , following (5), (6) can be written in the form δg. The fourth term in form $g\cup_1 g$ where $g_i=0,\ i<n+1$ and $g_i=\pi_i,\ i>n$.

Therefore the stasheff relation (1) takes the form $\delta g+g\cup_1 g$, and hence g is twisted cochain.

Thus every A_∞-structure from $(A,\pi_i)(\infty)$ defines a twisted cochain for Hochscihld complex C∞ (A, A). The inverse is true that is every twisted cochain defines A_∞- structure.

To complete the proof we must show that any two extension of A_∞-algebra are equivalent if and only if every equivalent result coincide with its twisted cochain.

From theorem 4.2 and definition 3.3 we get the following assertion.

Theorem (4.3):

If $H^2(C_\infty$ (A, A))=0, then any structure of extension of a fixed A_∞-algebra is trivial.

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