Characterizations of rectifying, normal and osculating curves in three dimensional compact Lie groups

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Abstract: Position vector of a curve provides us some advantages in mechanics, kinematics and differential geometry for characterizations of curves. So, some authors [1, 4, 5, 6] have studied curves whose position vectors always lie their rectifying, normal and osculating plane, respectively. In this paper, we study the rectifying, normal and osculating curves in a three dimensional compact Lie group G with a bi-invariant metric. We give some new characterizations for these curves. Moreover, we obtain necessary and sufficient conditions for them using their harmonic curvature functions.

[Zehra Bozkurt, İsmail Gök, O. Zeki Okuyucu, F. Nejat Ekmekci. **Characterizations of rectifying, normal and osculating curves in three dimensional compact Lie groups.** *Life Sci J* 2013;10(3):819-823] (ISSN: 1097-8135).http://www.lifesciencesite.com. 123

Keywords: Primary 53C40; Secondary 22E15; Rectifying curves, normal curves, osculating curves, Lie groups

1. Introduction

Curves theory has studied in Euclidean 3-space $\mathbb{E}^3$ for a long time. To examine of the curves theory with the help of Frenet frame {T, N, B} provides us the convenience. The planes which is spanned by {T, B}, {N, B} and {T, N} are known as the rectifying, normal and osculating plane, respectively. A curve is called a osculating curve if its position vector always lies its osculating plane and called a normal curve if its position vector always lies its normal plane. Besides, the notion of rectifying curves was introduced by B. Y. Chen in [1]. He defined a curve called rectifying curve whose position vector always lies its rectifying plane. Rectifying curves and Darboux vectors play an important role in mechanics, kinematics and differential geometry.

So, lots of authors [1, 4, 5, 6] have studied the position vector of the curve which always lies its rectifying, normal or osculating plane in Euclidean and Lorentzian space. According to our opinion, position vector of a curve should be investigated in three dimensional Lie groups. Because we know that Lie groups play an enormous role in modern geometry. Various "geometries" by specifying an appropriate transformation group that leaves certain geometric properties invariant. Thus, Euclidean geometry corresponds to the choice of the group E(3) of distance-preserving transformations of the

Euclidean space $\mathbb{E}^3$ conformal geometry corresponds to enlarging the group to the conformal group, whereas in projective geometry one is interested in the properties invariant under the projective group. This idea later led to the notion of a G-structure, where G is a Lie group of "local" symmetries of a manifold. On a "global" level, whenever a Lie group acts on a geometric object, such as a Riemannian or a symplectic manifold, this action provides a measure of rigidity and yields a rich algebraic structure. The presence of continuous symmetries expressed via a Lie group action on a manifold places strong constraints on its geometry and facilitates analysis on the manifold. Linear actions of Lie groups are especially important and are studied in representation theory (see for detail in [8]).

In this paper, we define the rectifying, normal and osculating curves in a three dimensional compact Lie group G with a bi-invariant metric. Moreover, we give some new characterizations for these curves

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and D be the Levi-Civita connection of Lie group G. If g denotes the Lie algebra of G then we know that g is issomorphic to $T_e G$ where e is neutral element of G. If $\langle , \rangle$ is a bi-invariant metric on G, then we have

$$\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle \qquad (2.1)$$

and if X and Y are left invariant then

$$D_X Y = \frac{1}{2}[X,Y] \qquad (2.2)$$

for all X,Y and $Z \in$ g.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of g. In this case, we write that any two vector fields W and

Z along the curve α as $W=\sum_{i=1}^{n} w_i X_i$ and $Z=\sum_{i=1}^{n} z_i X_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also, the Lie bracket of two vector fields W and Z is given

$$[W,Z]=\sum_{i=1}^{n} w_i z_i [X_i, X_j]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha'}W$ is given as follows

$$D_{\alpha'}W = W' = \dot{W} + \frac{1}{2}[T,W] \qquad (2.3)$$

where $T=\alpha'$ and $\dot{W}=\sum_{i=1}^{n} \dot{w}_i X_i$ or $\dot{W}=\sum_{i=1}^{n} \frac{dw}{dt} X_i$. Note that if W is the restriction of a left-invariant vector field to the curve α, then $\dot{W}=0$ (see [2]).

Let G be a three dimensional Lie group and (T,N,B,κ,τ) denote the Frenet apparatus of the curve α, and calculate $\kappa=\|\dot{T}\|$.

Definition 1. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve in three dimensional Lie group G with the Frenet apparatus (T,N,B,κ,τ) then

$$\tau_G = \frac{1}{2}\langle [T,N],B\rangle \qquad (2.4)$$

or

$$\tau_G = \frac{1}{2\kappa^2\tau}\left\langle \ddot{T}, \left[T,\dot{T}\right]\right\rangle + \frac{1}{4\kappa^2\tau}\left\|\left[T,\dot{T}\right]\right\|^2$$

(see [3]).

Definition 2. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in three dimensional compact Lie group G with the Frenet apparatus $\{T,N,B,\kappa,\tau\}$. Then the harmonic curvature function of the curve α is defined by

$$H = \frac{\tau - \tau_G}{\kappa} \qquad (2.5)$$

(see [7]).

Theorem 1. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T,N,B,κ,τ). The curve α is a general helix (its tangent vector field T makes a constant angle with a left-invariant vector field X) if and only if the harmonic curvature function of the curve α is a constant function (see [3]).

Proposition 1. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in three dimensional compact Lie group G with the Frenet apparatus $\{T,N,B,\kappa,\tau,\tau_G\}$. Then the following equalities

$$[T,N] = \langle [T,N],B\rangle B = 2\tau_G B,$$
$$[T,B] = \langle [T,B],N\rangle N = -2\tau_G N$$

hold (see [7]).

Proposition 2. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in three dimensional compact Lie group G with the Frenet apparatus $\{T,N,B,\kappa,\tau,\tau_G\}$. Then the Frenet-Serret formulae in Lie group G is given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau-\tau_G \\ 0 & -(\tau-\tau_G) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \quad (2.6)$$

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve with the Frenet apparatus $\{T,N,B,\kappa,\tau\}$. By using the Eq. (2.3) we have that the covariant derivative of the tangent vector T is

$$T' = \kappa N.$$

Since $N' \in sp\{T,N,B\}$ we have

$$N' = b_1 T + b_2 N + b_3 B$$

and thus

$$b_1 = \langle N',T\rangle = -\langle T',N\rangle = -\kappa,$$
$$b_2 = \langle N',N\rangle = 0$$

then using the Eq. (2.3) and Proposition 2, we get

$$\begin{aligned} b_3 &= \langle N',B\rangle \\ &= \left\langle \dot{N} + \frac{1}{2}[N,T], B\right\rangle \\ &= \left\langle \dot{N}, B\right\rangle + \frac{1}{2}\langle [N,T],B\rangle \\ &= \tau - \tau_G \end{aligned}$$

therefore we can write

$$N' = -\kappa T + (\tau - \tau_G)B$$

and using the similar method we obtain

$$B' = -(\tau - \tau_G)N.$$

This comletes the proof.

Corollary 1. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in three dimensional compact Lie group G with the Frenet apparatus

$\{T, N, B, \kappa, \tau, \tau_G\}$. Then $\tau = \tau_G$ if and only if binormal vector of the curve α is a constant vector field.

Proof. Let τ be equal to τ_G. Then, by using the Eq. (2.6) we have $B' = 0$ and then B is a constant vector field.

Conversely, let B be constant vector field. So, we have $B' = 0$ or using the Frenet formulas in three dimensional compact Lie group G, we obtain $\tau = \tau_G$. This completes the proof.

3. Rectifying, normal and osculating curves in a three dimensional Lie group

In this section, we define rectifying, normal and osculating curves and give some characterizations of these curves in three dimensional compact Lie group G.

Definition 3. The notion of the *rectifying curve* is introduced in [1, 4] as space curve whose position vector always lies in its rectifying plane. Therefore, the position vector with respect to some chosen origin, of a rectifying curve in $\mathbb{R}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \eta(s)B(s) \qquad (3.1)$$

where $\lambda(s)$ and $\eta(s)$ are arbitrary differentiable functions in terms of the arc-length parameter $s \in I \subset \mathbb{R}$.

Definition 4. The notion of the *normal curve* is introduced in [6] as space curve whose position vector always lies in its normal plane. Therefore, the position vector with respect to some chosen origin, of a rectifying curve in $\mathbb{R}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)N(s) + \eta(s)B(s) \qquad (3.2)$$

where $\lambda(s)$ and $\eta(s)$ are arbitrary differentiable functions in terms of the arc-length parameter $s \in I \subset \mathbb{R}$.

Definition 5. The notion of the *osculating curve* is introduced in [5] as space curve whose position vector always lies in its osculating plane. Therefore, the position vector with respect to some chosen origin, of a rectifying curve in $\mathbb{R}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \eta(s)N(s) \qquad (3.3)$$

where $\lambda(s)$ and $\eta(s)$ are arbitrary differentiable functions in terms of the arc-length parameter $s \in I \subset \mathbb{R}$.

Definition 6. A curve α in 3-dimensional compact Lie group G is a *Bertrand curve* if there exists a special curve β in 3-dimensional compact Lie group G such that principal normal vector field of the curve α is linearly dependent normal vector field of the curve β at corresponding point under ψ which is bijection from α to β. In this case, β is called the *Bertrand partner curve* of α and (α, β) is called *Bertrand curve couple* in 3-dimensional Lie group G.

(see [9]) ·

Theorem 2. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Bertrand curve in 3-dimensional compact Lie group G with the Frenet apparatus $\{T, N, B, \kappa, \tau, \tau_G\}$. Then, α satisfy the following equality

$$\lambda\kappa(s) + \mu\kappa(s)H(s) = 1, \text{ for all } s \in I \quad (3.4)$$

where λ, μ are constants and H is the harmonic curvature function of the curve α (see [9]).

3.1. Rectifying curves in a three dimensional Lie group

Theorem 3. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in 3-dimensional compact Lie group G with the Frenet apparatus $\{T, N, B, \kappa, \tau, \tau_G\}$. Then the curve α is a rectifying curve if and only if the harmonic function H is a linear function, that is,

$$H(s) = as + b, \ a \neq 0 \qquad (3.5)$$

where a and b are constant.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a rectifying curve in 3-dimensional compact Lie group G. Without loss of generality, we may assume that α is a parametrized curve. Then by using the Definition 3 we have

$$\alpha(s) = \lambda(s)T(s) + \eta(s)B(s) \qquad (3.6)$$

where $\lambda(s)$ and $\eta(s)$ are arbitrary differentiable functions in arclength parameter $s \in I \subset \mathbb{R}$. Differentiating Eq. (3.6) with respect to s and using the Eq. (2.6), we get

$$T(s) = \lambda'(s)T(s) + (\lambda(s)\kappa(s) - \eta(s)(\tau(s) - \tau_G(s))N(s) + \eta'(s)B(s).$$

It follows that

$$\begin{cases} \lambda'(s) = 1 \\ \lambda(s)\kappa(s) - \eta(s)(\tau(s) - \tau_G(s)) = 0 \\ \eta'(s) = 0 \end{cases} \quad (3.7)$$

and thus

$$\lambda(s) = s + c_1 \text{ and } \eta(s) = c_2 \qquad (3.8)$$

or using the equations (3.7) and (3.8), we obtain

$$\frac{\tau(s)-\tau_G(s)}{\kappa}=\frac{s+c_1}{c_2}, \tag{3.9}$$

where c_1 and c_2 are real constants.

Conversely, suppose that H is a linear function with $\kappa>0$, that is

$$\frac{\tau(s)-\tau_G(s)}{\kappa}=\frac{s+c_1}{c_2}$$

where c_1 and c_2 are real constants. Let us consider the vector field $X\in M\subset G$ given as

$$X(s)=\alpha(s)-(s+c_1)T(s)-c_2B(s) \tag{3.10}.$$

From Eq. (3.10) we can easily find that $X'(s)=0$. Therefore α is a rectifying curve. This completes the proof.

Corollary 2. Let $\alpha : I\subset \mathbb{R}\to G$ be a rectifying curve in 3-dimensional compact Lie group G. Then α can not be a general helix.

Proof. It is obvious from Theorem 3.

Corollary 3. Let $\alpha : I\subset \mathbb{R}\to G$ be a rectifying curve in 3-dimensional compact Lie group G. Then $\beta : I^*\subset \mathbb{R}\to G$ is Bertrand mate of the curve α if and only if α has the curvature as $\kappa(s)=\dfrac{1}{s}$.

Proof. Let $\alpha : I\subset \mathbb{R}\to G$ be a rectifying curve in 3-dimensional compact Lie group G. Then using the Theorem 3, we have

$$H(s)=\frac{\tau(s)-\tau_G(s)}{\kappa}=\frac{s+c_1}{c_2}$$

it follows that

$$c_1\kappa+c_2\kappa H=\kappa s \tag{3.11}$$

Also, using the Theorem 2 we can easily see that

$$\kappa(s)=\frac{1}{s}.$$

Conversely, we assume that α has the curvature as $\kappa(s)=\dfrac{1}{s}$. Then we can easily see that the curve α is a Bertrand curve. Which completes the proof.

3.2. Normal curves in a three dimensional Lie group

Theorem 4. Let $\alpha : I\subset \mathbb{R}\to G$ be an arc-lengthed curve in 3-dimensional compact Lie group G with the Frenet apparatus $\{T,N,B,\kappa,\tau,\tau_G\}$. Then α is a normal curve if and only if

$$H+[\kappa H(\frac{1}{\kappa})']'=0 \tag{3.12}$$

for all $s\in I$.

Proof. Let $\alpha : I\subset \mathbb{R}\to G$ be a normal curve in 3-dimensional compact Lie group G. Without loss of generality, we may assume that α is a parametrized curve. Then by using the Definition 4 we have

$$\alpha(s)=\lambda(s)N(s)+\eta(s)B(s) \tag{3.13}$$

where $\lambda(s)$ and $\eta(s)$ are arbitrary differentiable functions in arclength parameter $s\in I\subset \mathbb{R}$. Differentiating Eq. (3.13) with respect to s and using the Eq. (2.6), we get

$$T(s)=-\lambda(s)\kappa(s)T(s)+(\lambda'(s)-\eta(s)(\tau(s)$$
$$-\tau_G(s))N(s)+(\eta'(s)+\lambda'(s)(\tau(s)-\tau_G(s)))B(s).$$

It follows that

$$\begin{cases}-\lambda(s)\kappa(s)=1\\ \lambda'(s)-\eta(s)(\tau(s)-\tau_G(s))=0\\ \eta'(s)+\lambda'(s)(\tau(s)-\tau_G(s))=0\end{cases} \tag{3.14}$$

and thus

$$\begin{cases}\lambda(s)=-\dfrac{1}{\kappa(s)}\\ \eta(s)=\dfrac{1}{\tau(s)-\tau_G(s)}(\dfrac{1}{\kappa(s)})'\end{cases} \tag{3.15}$$

Also, from third equation in Eq. (3.14), we obtain that

$$\frac{\tau(s)-\tau_G(s)}{\kappa(s)}+[(\frac{1}{\kappa(s)})'\frac{1}{\tau(s)-\tau_G(s)}]'=0$$

or

$$(\frac{1}{\kappa(s)})'^2$$
$$+[(\frac{1}{\kappa(s)})'\frac{1}{\tau(s)-\tau_G(s)}]^2=\text{const.} \tag{3.16}$$

Conversely, suppose that $\alpha : I\subset \mathbb{R}\to G$ is a curve in 3-dimensional compact Lie group G with $\kappa>0$ such that $H+[\kappa H(\frac{1}{\kappa})']'=0$. Let us consider the vector field $X\in M\subset G$ given as

$$X(s)=\alpha(s)+\frac{1}{\kappa(s)}N(s)$$
$$-\frac{1}{\tau(s)-\tau_G(s)}(\frac{1}{\kappa(s)})'B(s). \tag{3.17}$$

From Eq. (3.17) we easily find that $X'(s)=0$.
Therefore α is congruent to a normal curve.

Corollary 4. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in 3-dimensional compact Lie group G with the Frenet apparatus $\{T, N, B, \kappa, \tau, \tau_G\}$. Then α is a normal curve if and only if

$$H = ce^{\int \frac{1-(\ln \kappa)''}{(\ln \kappa)'} ds}$$

where c is constant.

Proof. It is obvious with the help of Eq. (3.12).

3.3. Osculating curves in a three dimensional Lie group

Theorem 5. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve in 3-dimensional compact Lie group G with the Frenet apparatus $\{T, N, B, \kappa, \tau, \tau_G\}$. Then α is a osculating curve if and only if α is a geodesic or binormal vector B of the curve α is constant.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a osculating curve in 3-dimensional compact Lie group G. Without loss of generality, we may assume that α is a parametrized curve. Then by using the Definition 5 we have

$$\alpha(s) = \lambda(s)T(s) + \eta(s)N(s) \qquad (3.18)$$

for some functions $\lambda(s)$ and $\eta(s)$. Differentiating Eq. (3.18) with respect to s and using the Eq. (2.6), we get

$$T(s) = (\lambda'(s) - \eta(s)\kappa(s))T(s) + (\eta'(s) + \lambda(s)\kappa(s))N(s) + (\eta(s)(\tau(s) - \tau_G(s))B(s).$$

It follows that

$$\begin{cases} \lambda'(s) - \eta(s)\kappa(s) = 1 \\ \eta'(s) + \lambda(s)\kappa(s) = 0 \\ \eta(s)(\tau(s) - \tau_G(s)) = 0 \end{cases} \qquad (3.19)$$

and thus from third equation in Eq. (3.19) we get

$$\eta(s) = 0 \text{ and } \tau(s) - \tau_G(s) = 0 \qquad (3.20)$$

If $\eta(s)$ is equal to zero then from Eq. (3.19) we have $\lambda(s)\kappa(s) = 0$ so $\lambda(s) = 0$ or $\kappa(s) = 0$. From first equation in Eq. (3.19) we get $\lambda(s) \neq 0$. Thus $\kappa(s) = 0$ this implies that α is a geodesic curve.

Conversely, suppose that $\alpha : I \subset \mathbb{R} \to G$ is a curve in Lie group G with $\kappa > 0$ such that $H + [\kappa H(\frac{1}{\kappa})']' = 0$. Let us consider the vector field $X \in M \subset G$ given as

$$X(s) = \alpha(s) + \frac{1}{\kappa(s)}N(s) - \frac{1}{\tau(s) - \tau_G(s)}\left(\frac{1}{\kappa(s)}\right)'B(s). \qquad (3.21)$$

From Eq. (3.21) we easily find that $X'(s) = 0$.
Therefore α is congruent to a osculating curve.
If $\tau(s) - \tau_G(s) = 0$ then via Proposition 2, the binormal vector field B of the curve α is constant.

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12/21/2012