New exact solutions of the generalized fractional Zakharov-Kuznetsov equations

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Abstract: In this paper, the new extended trial equation method is used to solve nonlinear fractional partial differential equations. Based on the fractional derivative and traveling wave transformation, the fractional partial differential equation is turned into the nonlinear non-fractional ordinary differential equation. From here, we apply the new extended trial equation method, which is developed by the complete discrimination system for polynomial method, to this nonlinear non-fractional ordinary differential equation. As a result, some new exact solutions to this nonlinear problem are successfully constructed such as the elliptic integral function F, Π solutions.

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1. Introduction

In the recent years, the fractional differential equations play an outstanding role in physics, applied mathematics, chemistry and engineering. In some sense fractional differential equations could represent various real-life problems. However, the effective and general method for solving them has not been found even in the most useful works. Also, remarkable progress has been become in the determination of the approximate solutions of fractional nonlinear partial differential equations [1, 3]. So, some new methods in finding of the exact solutions to the fractional differential equations have been constructed. The exact solutions of these problems, when they exist, are very important in the understanding of most fractional nonlinear physical phenomena. There are important mathematical techniques which can be constituted the exact solutions for time fractional nonlinear differential equations [4-5]. Single kink soliton solutions, multiple-soliton solutions, compacton-like solutions, singular solitons and other solutions have been found by use of these approaches. Apart from all these, some new exact solutions have been obtained by using the trial equation methods. Some of them are elliptic integral functions F, E and Π, Jacobi elliptic function solutions.

In Section 2, primarily we have given some definitions and properties of the fractional calculus theory and also produce a new trial equation method for fractional nonlinear evolution equations with higher order nonlinearity. The power of this approach shows that it can be applied to different nonlinear physical problems. In Section 3, as an application, we have solved a fractional nonlinear partial differential equation such as the generalized fractional Zakharov-Kuznetsov equation [6]

$$\frac{\partial^\alpha u}{\partial t^\alpha}+a\left(u^m\right)_x+b\left(u^n\right)_{xxx}+c\left(u^r\right)_{yyx}=0,\quad t>0,\quad 0<\alpha\le 1, \tag{1}$$

where a, b, c, m, n, and r are real valued constants. Also, Eq. (1) is a mathematical model that governs the behavior of weakly nonlinear ion acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. Although there are a lot of studies for the classical generalized Zakharov-Kuznetsov equation and some favorable results have been reported, it is seen that detailed studies of the nonlinear fractional differential equation are only the beginning. Using the new trial equation method, we have found some new exact solutions of the fractional nonlinear physical problem.

The purpose of this paper is to obtain exact solutions of the generalized fractional Zakharov-Kuznetsov equations by new extended trial equation method, and to determine the accuracy of new extended trial equation method in solving these kinds of problems.

2. Preliminaries and the new extended trial equation method

In this section of the paper, we give several definitions and properties about the theory of fractional calculus. In order to obtain more detailed information with respect to the fractional calculus, we indicate the reader to [1-3]. Apart from this, we shortly examine the modified Riemann-Liouville fractional derivative defined by Jumarie [7-9]. Let $f:[0,1]\to\Re$ be a continuous function and $\alpha\in(0,1)$. The Jumarie's modified fractional derivative of order α and the function f may be defined by expression in [10] as follows:

$$D_x^\alpha f(x)=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_0^x (x-\xi)^{-\alpha-1}\left[f(\xi)-f(0)\right]d\xi, & \alpha<0;\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi, & 0<\alpha<1;\\ \left(f^{(n)}(\xi)\right)^{\alpha-n} & ,n\le\alpha\le n+1, n\ge 1\end{cases} \quad (2)$$

In addition to this definition, we can give some basic properties of the modified fractional Riemann-Liouville derivative. Some useful formulas are given as

$$D_x^\alpha k=0,$$

$$D_x^\alpha x^\mu=\begin{cases}0, & \mu\le\alpha-1\\ \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-x}, & \mu>\alpha-1\end{cases}$$

Some new trial equation methods were defined in literature [11-21]. In this paper, a new approach to the trial equation method has been given. In order to apply this method to fractional nonlinear partial differential equations, we consider the following steps.

***Step* 1.** We take the space-time fractional partial differential equation in two independent variables and a dependent variable u

$$P\left(u, D_t^\alpha u, u_x, u_{xx}, u_{xxx}, \cdots\right)=0, \quad (3)$$

and use the wave transformation

$$u(x,t)=u(\eta),\quad \eta=x-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \quad (4)$$

where $\lambda\neq 0$. Substituting Eq. (4) into Eq. (3) yields a nonlinear ordinary differential equation

$$N\left(u,u',u'',u''',\cdots\right)=0. \quad (5)$$

***Step* 2.** Take the trial equations as follows:

$$(u')^2=\frac{F(u)}{G(u)}=\frac{\sum_{i=0}^{k}a_i u^i}{\sum_{j=0}^{l}b_j u^j} \quad (6)$$

and

$$u''=\frac{F'(u)G(u)-F(u)G'(u)}{2G^2(u)}, \quad (7)$$

where $F(u)$ and $G(u)$ are polynomials. Substituting above equations into Eq. (5) yields an algebraic equation of polynomial $\Omega(u)$ of u:

$$\Omega(u)=\rho_s u^s+\cdots+\rho_1 u+\rho_0=0. \quad (8)$$

According to the balance principle, we can get a relation between the values of k and l. Then, we can determine some values of k and l.

***Step* 3.** Let the coefficients of $\Omega(u)$ all be zero will yield a system of algebraic equations:

$$\rho_i=0,\qquad i=0,\cdots,s \quad (9)$$

Solving this system, we will specify the values of $a_0,\cdots,a_k$ and $b_0,\cdots,b_l$.

***Step* 4.** Reduce Eq. (6) to the elementary integral form

$$\pm(\mu-\mu_0)=\int\sqrt{\frac{G(u)}{F(u)}}du \quad (10)$$

Using a complete discrimination system for polynomial to classify the roots of $F(u)$, we solve Eq. (10) with the help of MATHEMATICA and classify the exact solutions to Eq. (5). Otherwise, we can write the traveling wave solutions to Eq. (3), respectively.

3. Application to the generalized fractional Zakharov-Kuznetsov equation

Taking $m=n=r$ in Eq. (1), we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha}+a\left(u^n\right)_x+b\left(u^n\right)_{xxx}+c\left(u^n\right)_{yyx}=0. \quad (11)$$

In the case of $\alpha=1$, Eq. (11) can be reduced to the classical nonlinear dispersive KdV equation. Using a variety of methods [22-23], many authors have tried to find the exact solutions of this equation. Compactons, periodic and solitary traveling plane waves solutions of this equation have been found in the literature.

In order to construct the traveling wave solutions of Eq. (11), we perform the new wave transformation $u(x,y,t)=u(\eta)$, $\eta=x+y-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}$ where λ is a constant to be determined later. Then, integrating this equation with respect to η and setting the integration constant to zero, we have

$$(b+c)\left(u^n\right)''+a u^n-\lambda u=0. \quad (12)$$

By use of the transformation

$$u=v^{\frac{1}{n-1}}, \quad (13)$$

Eq. (12) can be transformed into the equation

$$(b+c)n(n-1)vv''+(b+c)n(v')^2-\lambda(n-1)^2 v+a(n-1)^2 v^2=0. \quad (14)$$

Substituting Eqs. (6) and (7) into Eq. (14) and using balance principle yield

$$k=l+2.$$

By use of the solution procedure, we obtain the following results:

Case 1

If we take $l=0$ and $k=2$, then

$$(v')^2=\frac{a_0+a_1 v+a_2 v^2}{b_0},\quad v''=\frac{a_1+2a_2 v}{2b_0}, \quad (15)$$

where $a_2 \neq 0$, $b_0 \neq 0$. Thus, we have a system of algebraic equations from the coefficients of polynomial of u. Solving the system of nonlinear algebraic equation, we get

$$a_0 = 0, \quad a_1 = a_1, \quad a_2 = -\frac{a(n-1)^2 b_0}{(b+c)n^2},$$
$$b_0 = b_0, \quad \lambda = \frac{(b+c)(n^2+n)a_1}{2(n-1)^2 b_0}. \tag{16}$$

Substituting these coefficients into Eq. (10), we have

$$\pm(\mu-\mu_0) = \sqrt{-\frac{(b+c)n^2}{a(n-1)^2}} \int \frac{du}{\sqrt{u^2 - \frac{(b+c)n^2 a_1}{a(n-1)^2 b_0} u}}. \tag{17}$$

Integrating Eq. (17), we produce the solutions to the Eq. (11) as follows:

$$\pm(\mu-\mu_0) = A \ln|u-\alpha_1|, \tag{18}$$

$$\pm(\mu-\mu_0) = 2A \ln\left|\sqrt{u-\alpha_1} + \sqrt{u-\alpha_2}\right|, \tag{19}$$

where $A = \sqrt{-\frac{(b+c)n^2}{a(n-1)^2}}$. Also α_1, α_2 are the roots of the polynomial equation

$$u^2 + \frac{a_1}{a_2}u + \frac{a_0}{a_2} = 0. \tag{20}$$

Therefore, we find the exact solutions

$$u(x,y,t) = \left[\exp\left[\frac{1}{A}\left(x+y-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}-\eta_0\right)\right]+\alpha_1\right]^{\frac{1}{n-1}}, \tag{21}$$

$$u(x,y,t) = \left[\frac{\exp\left[\frac{1}{A}\left(x+y-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}-\eta_0\right)\right]+(\alpha_1-\alpha_2)^2\exp\left[\frac{-1}{A}\left(x+y-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}-\eta_0\right)\right]+2(\alpha_1+\alpha_2)}{4}\right]^{\frac{1}{n-1}}. \tag{22}$$

If we take $\eta_0 = \alpha_2 = 0$ and $\alpha_1 = 1$, then the solutions (21) and (22) can reduce to single kink and hyperbolic function solutions respectively,

$$u(x,y,t) = \left(\exp\left[B\left(x+y-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right]+1\right)^{\frac{1}{n-1}}, \tag{23}$$

$$u(x,y,t) = \left(\frac{1}{2}\left(\cosh\left[B\left(x+y-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right]+1\right)\right)^{\frac{1}{n-1}}, \tag{24}$$

where $B = 1/A$. Here, B is the inverse width of the solitons.

Remark 1. The solutions (23) and (24) found by using the new trial equation method for Eq. (11) have been checked by Mathematica. To our knowledge, the hyperbolic function and single kink solutions, that we find in this paper, has not been shown in the previous literature. These results are new traveling wave solutions of Eq. (11).

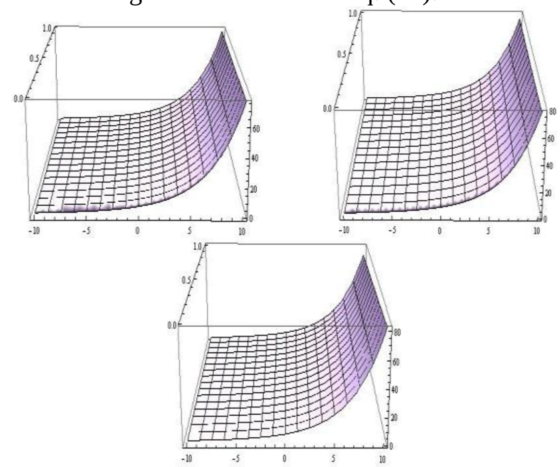

Figure 1. Solution of (21) is shown at $B=\frac{1}{7}$, $a=-0.5$, $b=c=1$, $n=1.4$, $\lambda=0.29$ and $\alpha=0.01, 0.055, 0.85$.

Case 2

If we take $l=1$ and $k=3$, then

$$(v')^2 = \frac{a_0 + a_1 v + a_2 v^2 + a_3 v^3}{b_0 + b_1 v},$$
$$v'' = \frac{(b_0+b_1 v)(a_1 + 2a_2 v + 3a_3 v^2) - b_1(a_0 + a_1 v + a_2 v^2 + a_3 v^3)}{2(b_0+b_1 v)^2}, \tag{25}$$

where $a_3 \neq 0$, $b_1 \neq 0$. Respectively, solving the system of nonlinear algebraic equations yields

$$a_0 = 0, \quad a_1 = -\frac{a(n-1)^2 b_0\left(n^2(b+c)a_2 + a(n-1)^2 b_0\right)}{(b+c)^2 n^4 a_3}, \tag{26}$$
$$a_2 = a_2, \quad a_3 = a_3, \quad b_0 = b_0, \quad b_1 = -\frac{(b+c)n^2 a_3}{a(n-1)^2},$$
$$\lambda = -\frac{a(n+1)\left(n^2(b+c)a_2 + a(n-1)^2 b_0\right)}{2(b+c)n^3 a_3}.$$

Substituting these results into Eq. (10), we have

$$\pm(\mu-\mu_0) = \sqrt{-\frac{(b+c)n^2}{a(n-1)^2}} \int \sqrt{\frac{u - \frac{a(n-1)^2 b_0}{(b+c)n^2 a_3}}{u^3 + \frac{a_2}{a_3}u^2 - \frac{a(n-1)^2 b_0\left(n^2(b+c)a_2 + a(n-1)^2 b_0\right)}{(b+c)n^4 a_3^2}u}}\, du. \tag{27}$$

Integrating Eq. (27), we obtain the new solutions to the Eq. (11) as follows:

$$\pm(\mu-\mu_0) = 2A\left(\ln\left|\sqrt{\frac{b_0+b_1 u}{b_1}} + \sqrt{u-\alpha_1}\right| - \sqrt{\frac{b_0+b_1 u}{b_1(u-\alpha_1)}}\right), \tag{28}$$

$$\pm(\mu-\mu_0) = \frac{2A}{\sqrt{b_1(\alpha_2-\alpha_1)}}\left\{\begin{array}{l}\sqrt{b_0+b_1\alpha_1}\arctan\left[\sqrt{\frac{(b_0+b_1\alpha_1)(u-\alpha_2)}{(b_0+b_1 u)(\alpha_2-\alpha_1)}}\right] \\ +\sqrt{\alpha_2-\alpha_1}\ln\left|\sqrt{\frac{b_0+b_1 u}{b_1}} + \sqrt{u-\alpha_2}\right|\end{array}\right\}, \tag{29}$$

$$\pm(\mu-\mu_0)=\frac{2A}{\sqrt{b_1(\alpha_1-\alpha_2)(b_0+b_1\alpha_3)}}\begin{pmatrix}(b_0+b_1\alpha_1)F(\varphi,l)+\\(\alpha_3-b_1\alpha_1)\Pi(\varphi,n,l)\end{pmatrix}, \tag{30}$$

where

$$A=\sqrt{-\frac{(b+c)n^2}{a(n-1)^2}},\qquad F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},$$

$$\Pi(\varphi,n,l)=\int_0^{\varphi}\frac{d\psi}{\left(1+n\sin^2\psi\right)\sqrt{1-l^2\sin^2\psi}} \tag{31}$$

and

$$\varphi=\arcsin\sqrt{\frac{(u-\alpha_3)(\alpha_2-\alpha_1)}{(u-\alpha_1)(\alpha_2-\alpha_3)}},\ n=\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2} \tag{32}$$

$$l^2=\frac{(b_0+b_1\alpha_1)(\alpha_3-\alpha_2)}{(b_0+b_1\alpha_3)(\alpha_1-\alpha_2)}$$

Also, the values of α_1, α_2 and α_3 are the roots of the polynomial equation

$$u^3+\frac{a_2}{a_3}u^2+\frac{a_1}{a_3}u+\frac{a_0}{a_3}=0. \tag{33}$$

4. Conclusions

In this article, the new extended trial equation method is applied successfully for solving the nonlinear fractional differential equations. We used it to obtain some soliton and elliptic function solutions to the generalized fractional Zakharov-Kuznetsov equation. The performance of this method is reliable and effective, and also this method gives more general solutions. We think that the new approach proposed in this paper can also be applied to other generalized fractional nonlinear differential equations.

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