Exact Solutions of Space-Time Dependent Korteweg-de Vries Equation by The Extended Unified Method

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Abstract: Recently the unified method for finding traveling wave solutions of nonlinear evolution equations was proposed by one of the authors. It was shown that, this method unifies all the methods being used to find these solutions. In this paper, we extend this method to find a class of formal exact solutions to Korteweg-de Vries (KdV) equation with space-time dependent coefficients. A new class of multiple-soliton or wave trains is obtained.
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1 Introduction

We consider the following evolution equation

$$f\left(x,t,u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x\partial t},\frac{\partial^2 u}{\partial x^2},\ldots,\frac{\partial^m u}{\partial x^m}\right)=0,\quad m\geq 1, \quad (1)$$

where f is a polynomial in its arguments. When Eq. (1) does not depend explicitly on x and t, it can be reduced to a subclass of ordinary differential equations by using the Lie groups for partial differential equations [1] or by using similarity transformations. Among these equations, the traveling wave has the form

$$g(u,u',u'',\ldots u^{(m)})=0,\, u'=\frac{du}{dz},\, z=x-ct, \quad (2)$$

which results due to the translation symmetry of (1). The Painleve' analysis is used to testing the integrability of partial differential equations, that was developed in [2]. Auto-B $\ddot{a}$ cklund transformation deals with the exact solutions that were obtained for integrable forms of (2) by truncating Painleve' expansion [3-9]. Recently auto-B $\ddot{a}$ cklund transformation that was extrapolated in [10-14] and the homogeneous balance method in [15-19] assert a solution for evolution equations with variable coefficients in the form

$$u(x,t)=\frac{\partial^{m-2}}{\partial x^{m-2}}(a(\phi)\phi_x)+u^{(0)}(x,t),$$

where ϕ is the base function.

2 Extended unified method

Explicit solutions of Eq. (2) are, in fact, particular solutions. In this respect, this solutions are mapped to other solutions that are given in terms of known elementary or special functions. Recently in [20] the class of these solutions were obtained by the generalized mapping method (GMM). This method generalizes the results as a polynomial or a rational function solutions. In the present paper, we extend this method to handle equations of type (1).

2.1 Polynomial solutions

In this section, we search for polynomial solutions of Eq. (2) in $C^S(\mathrm{R})$,

$$S=\{\phi:\mathrm{R}\times\mathrm{R}^+\to\mathrm{K},\, \phi_t^q=P_{k_1}^t(\phi),(\phi_x)^p=P_k^x(\phi)\},$$

$$P_{k_1}^t(\phi)=\sum_{i=0}^{k_1}b_i(x,t)\phi^i(x,t), P_k^x(\phi)=\sum_{i=0}^{k}c_i(x,t)\phi^i(x,t).$$ Indeed the set S contains elementary or elliptic functions for some particular values of q,p,k_1 and k. The mapping method asserts that there exists a positive integer n and a mapping

$$M:C^S(\mathrm{R})\to\Omega,\quad \Omega=\{v,v=\sum_{i=0}^{s}a_i(x,t)\phi^i,\phi\in S\}$$ such that $M(u)=P_n(\phi)$ and satisfies the properties

$$M(\alpha_1u_1+\alpha_2u_2)=\alpha_1M(u_1)+\alpha_2M(u_2),$$

$$M(u_1u_2)=M(u_1)M(u_2), M(u_t)=(M(u))_t, M(u_x)=(M(u))_x.$$

Thus M is a ring homomorphism that conserves differentiation. By the former conditions we find that, $M(u_t)=P^t_{(n-1+k_1)}(\phi)\in\Omega, M(u_x)=P^x_{(n-1+k)}(\phi)\in\Omega.$ By using the properties of M and the last results and as $f\equiv f(x,t,u,u_t,\ldots)$ is a polynomial in its arguments, we find that $M(f)$ is a polynomial and there exists $s_o\leq s$ such that $M(f)=P_{s_o}(\phi)\in\Omega$. It is worthy to notice that all these polynomials have different coefficients. More simply the mapping M assigns to u and f gives two auxiliary equations, the polynomials $P_n(\phi)$ and

$P_{s_o}(\phi)$ respectively. In case of Eq. (1) $s_o = n - m + mk$. The utility of the above presentation helps us to give arguments to the statements of the conditions in lemmas 2.1 and 2.2. Also, we think that it allows for constructing more generalization and it is more appropriate when (1) is a vector equation.

We substitute for $u, u_t, u_x, ..., \frac{\partial^m}{\partial x^m} u$ as polynomials in ϕ , so that the function f is a polynomial in ϕ, together with two auxiliary equations. In the applications we may write directly $u = P_n(\phi)$, and $f = P_{s_o}(\phi)$. From the previous analysis we may write

$$u = \sum_{i=0}^{n} a_i(x,t)\phi^i, \qquad (3)$$

where for instance we assume that $k_1 = k$, so that the auxiliary equations are

$$\phi_t = \sum_{i=0}^{k} b_i(x,t)\phi^i, \phi_x = \sum_{i=0}^{k} c_i(x,t)\phi^i, \quad (4)$$

together with the compatibility equation

$$\phi_{xt} = \phi_{tx}. \qquad (5)$$

We mention that solutions of (4) when exist, are elementary ($p = q = 1$). The case of elliptic solutions ($p = q = 2$) will be considered in a future work.

When substituting from (3) and (4) into (1) we find that it is transformed to $P_{s_o}^{(f)}(\phi) \equiv 0$ that gives rise to

$$\sum_{i=0}^{s_o} h_i(a_{r_o}(x,t), b_{r_1}(x,t), c_{r_2}(x,t), a_{r_o t}(x,t), a_{r_o x}(x,t), ...)\phi^i \equiv 0, \quad (6)$$
$$r_0 = 0,1,...,n \quad and \quad r_1, r_2 = 0,....,k,.$$

By equating the coefficients of $\phi^i, i = 0,1,...,s_o$ to zero, we get a set of $(s_o + 1)$ algebraic (or differential) equations, namely the principle equations, in the functions a_i, b_i and c_i . On the other hand the equations that result from (5) count: $2k - 1$, $k \geq 2$. We mention that these later unknown functions count: $n + 2k + 3$.

In Eq.(1), if $u^j u_x$ and $\frac{\partial^m}{\partial x^m} u$ are the highest nonlinear and the highest order derivative terms respectively, then we get the balancing condition as $s_o = nj + n + k - 1 = n - m + mk$. Thus by solving for n, we find that it depends on m, j and k. The last result and the number of compatibility equations namely $2k - 1$, $k \geq 2$ determine if the equations to be solved are over-determined or under-determined. The number of the determining equations, balances the number of unknowns, is over-determined or is under-determined when the difference, namely $(n - m + mk + 1) + (2k - 1) - (n + 2k + 3)$ is $0, > 0,$ or < 0 respectively. From this last conditions, we may determine a consistency condition that will be identified in the lemmas. In what follows necessary conditions for the existence of polynomial solutions will be stated.

Lemma 2.1. *For polynomial-solutions of (1) (as a polynomial in ϕ) to exist it is necessary that*

(i) $(m-1)(k-1)/j (:= n)$ is a positive integer

(ii) $m(k-1) - 3 \leq m$ when the equation (1) in the absence of x , and t passes the Painleve' test . Otherwise m is replaced by 2.

We notice that the first and the second conditions in lemma 2.1 are the balancing and the consistency conditions respectively. For details see [20].

2.2 The rational function solutions

Here, also we search for solutions of Eq. (1) in $C^S(\mathbb{R})$. For rational function-solutions of Eq.(1), we consider the space of functions $\Omega_R = \{v, v = P_n(\phi)/Q_r(\phi), \phi \in S\}$ and $Q_r(\phi)$ has no zeros in $\mathbb{K} \subset \mathbb{R}$. The definitions in the above and the GMM for rational function solutions assert that there exists a mapping

$$M_R : C^S(\mathbb{R}) \to \Omega_R, \quad M_R(u) = P_n(\phi)/Q_r(\phi), \quad \phi \in S.$$

The properties of these mapping are the same properties of the mapping $M(u)$ in section 2.1. By bearing in mind these properties and from (4), (5) we find that

$$M_R(u_t) = P_{1,(n-1+k+r)}(\phi)/Q_r^2(\phi), M_R(u_x) = P_{2,(n-1+k+r)}(\phi)/Q_r^{m+1}(\phi),$$

thus $M_R(\frac{\partial^i u}{\partial x^i}) \in \Omega_R, i = 1,...,m$. By using the properties of M_R and the last results, we get $M_R(f) \in \Omega_R$ and there exists $s_1 \leq s$ such that $M_R(f) = P_{s_1 R}(\phi)/Q_r^{m+1}(\phi)$. Indeed s_1 depends on n, r, k and also on m , where in the case mentioned in the above $s_1 = n - m + mk + mr$. Simply, we write

$$u = \sum_{i=0}^{n} a_i \phi^i / \sum_{i=0}^{r} d_i \phi^i. \qquad (7)$$

So that the Eq. (1) is transformed to $P_{s_1R}(\phi) \equiv 0$. Equivalently, the last identity becomes

$$\sum_{i=0}^{s_1} h_i(a_{r_0}(x,t), d_{r_2}(x,t), b_{r_1}(x,t), c_{r_3}(x,t), \ldots)\, \phi^i = 0, \quad (8).$$

$$r_0 = 0,\ldots,n, r_1, r_3 = 0,\ldots,k, \quad and \quad r_2 = 0,\ldots,r.$$

In (8), by equating the coefficients of $\phi^i, i = 0,1,\ldots,s_1$ to zero, we get a set of (s_1+1) equations that determine the functions a_i, b_i, c_i and d_i. We mention that these later functions count $n+2k+r+3$. By using the same assumptions on Eq. (1), as in section 2.1, the balancing condition is

$$\begin{cases} nj+n+k-1+r = n-m+mk+mr+r(j-(m+1)), & m+1<j \\ nj+r((m+1)-j) = n-m+mk+mr = s_1, & m+1>j \end{cases} \qquad (9)$$

Now by solving (9) for n, we find that it depends on m, j, r and k and, in both two cases, we get the same equation for $n-r$. Hereafter, we distinguish between the two cases mentioned in (9). From the last results and when $j < m+1$, the number of the determining equations, balances the number of unknowns, is over-determined or is under-determined when the difference, namely $(n-m+mk+rm+1)+(2k-1)-(n+2k+r+3)$ is $0, >0, or <0$ respectively. But when $j > m+1$ this difference is

$$(n-m+mk+rm+1+r(j-(m+1)))+$$
$$(2k-1)-(n+2k+r+3).$$

From these last conditions, we may determine the consistency condition that will be identified in the following Lemma.

Lemma 2.2. *For solitary wave-rational solutions of Eq. (2) to exist it is necessary that*

(i) $(m-1)(k-1)/j (:= n-r)$ *is an integer*

(ii) $r(m-1)+(k-1)m-3 \le m,\ j < m+1$ *or* $r(j-2)+(k-1)m-k-2 \le 2,\ j > m+1$, *in the case when Eq. (1) passes the Painleve' test. Otherwise* $r(m-1)+(k-1)m-k-2 \le 2, \quad j < m+1$ *or* $r(j-2)+(k-1)m-k-2 \le 2,\ j > m+1$.

For details see [20].

3 Exact solutions of space-time dependent KdV equation

We consider the following KdV equation with variable coefficients

$$u_t + f(x,t)u_{xxx} + g(x,t)uu_x = 0, t>0, x>0, \quad (10)$$

where f and g are arbitrary functions of x and t. For $x<0$, the solutions of Eq. (10) hold if we replace x by $|x|$ and assuming that $f(-x,t) = -f(x,t)$ and $g(-x,t) = -g(x,t)$. We mention that Eq. (10) describes the propagation of waves in a medium with space-time dependent dispersion and conviction. In fact, differential equations with variable coefficients may be of practical interests. Some exact solutions were obtained in Nirmala and Vedan [21] and E. Fan [12] when the coefficients in Eq. (10) are time dependent, namely $f(t)$ and $g(t)$. In these works, solutions were obtained when $f(t) = cg(t)$, where c is a constant. Under this condition Eq. (10) with time dependent coefficients can be transformed to a KdV equation with constant coefficients by using the transformations $\tau = \int g(t)dt$, $x = x$, $u = u$. In this case we obtain the well known solutions as soliton, solitary, or elliptic wave solutions.

3.1 The polynomial function solutions

In lemma 2.1, the consistency condition holds when $k = 2,3$ but it does not hold when $k \ge 4$. So that, only the cases $k = 2,3$ will be considered.

- First case: When $k = 2, n = 2$, by substituting into (3), (4) and (10), we get six principle equations. We mention that calculations are carried out by using MATHEMATICA where standard functions in calculus and algebra were only needed. The steps of computations are as follows;

Step 1. Solving the principle equations, where five of them are solved explicitly to

$$a_2(x,t) = -12h(x,t)c_2^2(x,t),$$

$$a_1(x,t) = -\frac{12}{5}(5h(x,t)c_1(x,t)c_2(x,t) +$$
$$c_2(x,t)h_x(x,t) + 5h(x,t)c_{2x}(x,t), \qquad (11)$$

together with explicit equations for $b_2(x,t)$, $b_1(x,t)$ and $b_0(x,t)$ (they are too lengthy to written here) where $h(x,t) = \dfrac{f(x,t)}{g(x,t)}$ and

$k(x,t)=\dfrac{1}{g(x,t)}$. It remains only one unsolved equation of the principle ones.

Step 2. We consider the compatibility equations that result from $\phi_{xt}=\phi_{tx}$ and they are given formally by;

$$b_0(x,t)c_1(x,t)-b_1(x,t)c_0(x,t)+c_{0t}(x,t)-b_{0x}(x,t)=0,$$

$$2b_0(x,t)c_2(x,t)-2b_2(x,t)c_0(x,t)+c_{1t}(x,t)-b_{1x}(x,t)=0, \quad (12)$$

$$-b_2(x,t)c_1(x,t)+b_1(x,t)c_2(x,t)+c_{2t}(x,t)-b_{2x}(x,t)=0.$$

To simplify the computations, we make the transformations

$$c_{2x}(x,t)=p(x,t)c_2(x,t), c_1(x,t)=-p(x,t)+C_1(x,t), \quad (13)$$

$$c_0(x,t)=\frac{-2C_{1x}(x,t)+C_1^2(x,t)+4C_0(x,t)}{4c_2(x,t)},$$

where $C_0(x,t), C_1(x,t)$ are arbitrary functions. To evaluate $a_0(x,t)$ the following steps are used.

i- Solve the last equation in (12) for a_{0x}

ii- Eliminate a_{0xx}, a_{0xxx}

iii- Substitute in the middle equation in (12) to get $a_0(x,t)$

iv- Calculate a_{0x} from the last step and identify it by a_{0x} from step (i), we get an equation in $C_0, C_{0x},\ldots$.

As the computations are too lengthy in the general case, we consider a power law functions $h(x,t)=h_0(t)x^q, k(x,t)=k_0(t)x^p$. Now to find C_0, we use the following steps of computations;

v- Solve the equation that result from (iv) in C_{0x}

vi- Substitute into the first equation in (12) and solve for C_{0t}. Thus (12) solved completely.

vii- Calculate C_{0tx} from (vi) and balance it with C_{0xt} from (v), we get the following algebraic equations

$$(125p^4-100p^3(-10+q)+p^2(1375+750q-270q^2)-2p(1250-2175q+315q^2+124q^3)+q(-4250+7325q-3622q^2+547q^3)+100(5p^2+4q(q-1)+p(9q-5))x^2C_0(x,t))), \quad (14)$$

or

$$(30+5p^2+p(25-9q)-22q+4q^2)=0, \quad (15)$$

or

$$-10+21q-14q^2+3q^3=0, \quad (16)$$

together with the remaining equation in the principle ones, which is too lengthly equation to be written

viii- Calculate $C_0(x,t)$ and balance $C_{0x}(x,t)$ with that one which computed from the step (v)

Case (1): Evaluate $C_0(x,t)$ from equation (14). By using the step (viii) and by solving the resultant equation from this step simultaneously with lengthly equation, we get a solution only when $p=2, q=5$ and

$h_0(t)=\dfrac{1}{6(h_1+\int\frac{1}{k_0(t)}dt)}$, where h_1 is a constant.

By solving the second auxiliary equation in (4), we get $\phi(x,t)$ as

$$\phi(x,t)=-\frac{2+s(t)+\log(x)+xC_1(x,t)(s(t)+\log(x))}{2xC_2(x,t)(s(t)+\log(x))}, \quad (17)$$

where $s(t)$ is an arbitrary function which can be calculated from the first auxiliary equation in (4).

By a direct calculations, we get the solution of (10) as

$$u(x,t)=-\frac{72x^3}{(h_1+\int\frac{1}{k_0(t)}dt)(6s_0+6\log(x)-11\log(h_1+\int\frac{1}{k_0(t)}dt))^2}, \quad (18)$$

where s_0 is a constant.

It is worth noticing that one can verify that the solution (given by (19)) satisfies (10).

Case (2): From (15) we have $q=p+3$ or $q=\dfrac{5(p+2)}{4}$.

(I) When $q=p+3$. By using lengthly equation, we get

$$h_0(t)=\frac{1}{2p(p^2-1)(h_1+\int\frac{1}{k_0(t)}dt)}\quad,\quad\text{and}$$

$$C_0(x,t)=-\frac{6(-21-4p+p^2)\pm\sqrt{15(-768+1024p+251p^2-1060p^3+481p^4)}}{600x^2}$$

By using the step (viii), we find that $p=-3$, or $p=\frac{16}{13}$.

- When $p=-3$, the solution of (10) is given by

$$u(x,t)=-\frac{s_0(s_0-2x(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{1}{4}})}{4x^2(h_1+\int\frac{1}{k_0(t)}dt)(s_0-x(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{1}{4}})^2}. \quad (19)$$

- When $p=\frac{16}{13}$, we get

$$u(x,t)=\frac{26s_0x^{\frac{29}{13}}(h_1s_0+s_0\int\frac{1}{k_0(t)}dt+13x^{\frac{2}{13}}(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{63}{464}})}{29(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{63}{232}}(13x^{\frac{2}{13}}+2s_0(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{401}{464}})^2}. \quad (20)$$

Again, the solutions (19) and (20) verify the equation (10).

(II) When $q=\frac{5(p+2)}{4}$. By the same way, we find that $p=\frac{2}{3}$, $h_0(t)=A_0$, and $C_0(x,t)=\frac{2}{9x^2}$.

Finally, we get

$$u(x,t)=-\frac{972A_0x^2}{(9(s_0+3x^{\frac{1}{3}})-28A_0\int\frac{1}{k_0(t)}dt)^2}, \quad (21)$$

where A_0 is a constant.

Case (3): From (3.8), we have $q=1$ or $q=2$.

(I) When $q=1$. By the same way, we find that $p=-2$,

$$h_0(t)=\frac{25}{42(h_1+\int\frac{1}{k_0(t)}dt)}\quad\text{and}$$

$$C_0(x,t)=\frac{9}{100x^2}.$$

Finally, we get

$$u(x,t)=-\frac{800x^{\frac{3}{5}}}{7(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{1}{5}}(4s_0+5x^{\frac{4}{5}}(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{2}{5}})^2}. \quad (22)$$

(II) When $q=2$. We find that $p=1$,

$$h_0(t)=\frac{A_0}{(h_1+\int\frac{1}{k_0(t)}dt)},\ C_0(x,t)=\frac{4}{25x^2}.$$

In this case, the solution of (10) is given by

$$u(x,t)=\frac{1}{50(h_1+\int\frac{1}{k_0(t)}dt)(5x^{\frac{3}{5}}+3s_0(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{3}{10}})^2}. \quad (23),$$

$$(25(5x^{\frac{8}{5}}+3xs_0(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{3}{10}})^2$$

$$-8A_0(650x^{\frac{6}{5}}-30x^{\frac{3}{5}}s_0(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{3}{10}}-9s_0^2(h_1+\int\frac{1}{k_0(t)}dt)^{\frac{3}{5}}).$$

The solutions (21), (22) and (23) verify the equation (10).

- **Second case:** When $k=3,n=4$, and by substituting into the equation (10), we get eleven principle equations. We solve eight ones of them to get $a_i(x,t),i=1,2,3,4$ and $b_j(x,t),j=0,1,2,3$. It remains three equations.

Now, the compatibility equations $\phi_{xt}=\phi_{tx}$ give rise to

$$b_0(x,t)c_1(x,t)-b_1(x,t)c_0(x,t)+c_{0t}(x,t)$$
$$-b_{0x}(x,t)=0,\quad 2b_0(x,t)c_2(x,t)+c_{1t}(x,t)$$
$$-2b_2(x,t)c_0(x,t)-b_{1x}(x,t)=0,$$
$$b_2(x,t)c_1(x,t)-b_1(x,t)c_2(x,t)+b_{2x}(x,t)-$$
$$3b_0(x,t)c_3(x,t)+3b_3(x,t)c_0(x,t)-$$
$$c_{2t}(x,t)=0\quad,\quad(24)$$
$$3b_3(x,t)c_1(x,t)-2b_1(x,t)c_3(x,t)-c_{3t}(x,t)+$$
$$b_{3x}(x,t)+b_3(x,t)c_2(x,t)-b_2(x,t)c_3(x,t)=0$$

To solve the equations (24) and those of the principle ones, we use the following transformations

$$c_{3x}(x,t)=Q(x,t)c_3(x,t),$$
$$c_1(x,t)=2C_1(x,t)+\frac{2c_2^2(x,t)}{3c_3(x,t)},$$
$$c_0(x,t)=C_0(x,t)-\frac{2C_2^3(x,t)+9C_{2x}(x,t)c_3(x,t)}{27c_3^2(x,t)}, \quad (25)$$

where $Q(x,t)$, $C_0(x,t)$, and $C_1(x,t)$ are arbitrary functions.

By following the same steps as we did in the first case and as the computations are too lengthy in the general case, we consider a power law functions $h(x,t) = h_0(t)x^q, k(x,t) = k_0(t)x^p$.

Now we solve (3.17) by using the steps (i)-(viii) and we get

$$Q(x,t) = -(2C_1(x,t) + \frac{2c_2^2(x,t)}{3c_3(x,t)}), \qquad (26)$$

$$C_0(x,t) = \pm \frac{5\sqrt{R(p,q)}}{2x^{\frac{3}{2}}\sqrt{6(25p^2+2q(5+q)-5p(5+3q))(10(1+p)-3q+30xC_1(x,t))c_3(x,t)}}, \qquad (27)$$
$$R(p,q) = 5q(-2+3q-q^2)(600+25p^3-75p^2(q-3) -620q+212q^2-24q^3+p(650-440q+74q^2)),$$

$$30 + 25p + 5p^2 - 22q - 9pq + 4q^2 = 0, \qquad (28)$$

and $C_1(x,t)$ is arbitrary, while the equation for $a_0(x,t)$ is too lengthy to be written here.

It remains to solve the three equations in the principle ones.

From (28) we have $q = p+3$ or $q = \frac{5(p+2)}{4}$.

(I) When $q = p+3$. By substituting about $q = p+3$ in the principle ones, two of them will be identically zero and the last one of them solves to

$$h_0(t) = \frac{25}{(48-25p-36p^2+37p^3)(h_1 + \int \frac{1}{k_0(t)}dt)}$$

where h_1 is a constant.

By solving the second auxiliary equation in (4), we get

$$\phi(x,t) = \frac{-c_2(x,t)}{3c_3(x,t)} \pm \frac{e^{\int (2C_1(x,t)+\frac{c_2^2(x,t)}{3c_3(x,t)})dx}}{\sqrt{A(t) - 2\int e^{2\int (2C_1(x,t)+\frac{c_2^2(x,t)}{3c_3(x,t)})dx} c_3(x,t)dx}}, \qquad (29)$$

where $A(t)$ is an arbitrary function.

It worth noticing that, in general, the condition $\phi_{xt} = \phi_{tx}$ is a necessary condition but not sufficient in this case. We have to assume that $C_1(x,t) = C_{10}(t)x^{s_1}$, and $c_3(x,t) = c_{30}(t)x^{s_2}$.

The sufficient condition for integrability of auxiliary equation(4) gives rise to

$$C_1(x,t) = \frac{-(3+p)}{10x}, c_2(x.t) = \sqrt{\frac{3(3+p-5s_2)}{10}}c_{30}(t). \qquad (30)$$

From the first auxiliary equation in (4), we evaluate $A(t)$. Finally, by a direct calculations we get the solution of (10) as

$$u(x,t) = \frac{12A_0(p-2)^3 x^{1+\frac{6p}{5}}(h_1+\int \frac{1}{k_0(t)}dt)^{-1+\alpha-\beta}(A_0(p-2)x^{\frac{p}{5}}(h_1+\int \frac{1}{k_0(t)}dt)^{\alpha-\beta}+20x^{\frac{2}{5}})}{(1+p)(48-73p+37p^2)(A_0(p-2)x^{\frac{p}{5}}(h_1+\int \frac{1}{k_0(t)}dt)^{\alpha-\beta}+10x^{\frac{2}{5}})^2},$$
$$\alpha = \frac{18(11+2p^2)}{5(48-25p-36p^2+37p^3)}, \beta = \frac{p(47+31p^2)}{5(48-25p-36p^2+37p^3)}, p \neq -1. \qquad (31)$$

We mention that, in (31) p is arbitrary and the solution (31) verifies the equation (10).

(II) When $q = \frac{5(2+p)}{4}$. We find that $p = 2/3$, $h(x,t) = A_0 x^{\frac{10}{3}}$, and $k(x,t) = k_0(t)x^{\frac{2}{3}}$.

The solution of (10) is given by

$$u(x,t) = -\frac{3888A_0x^2}{(9(s_0 - 6x^{\frac{1}{3}}) + 56A_0\int \frac{1}{k_0(t)}dt)^2} \qquad (32)$$

Again, the solution (32) verifies the equation (10).

It is Worthy to notice that the solutions that obtained in section 3.1 are rational functions in x^{s_i} for some s_i, but they are not rational in the function $\varphi(x,t)$.

3.2 The rational function solutions

Here, we seek for rational function-solutions of (10) where by using the condition (i) in lemma 2.2, we have two case; $r = n, k = 1$ and $(m-1)(k-1) = n - r$. We confine ourselves to the case when $n = r$.

In this case the equation (4) becomes as

$$\phi_t = b_1(x,t)\phi + b_0(x,t), \phi_x = c_1(x,t)\phi + c_0(x,t), \qquad (33)$$

together with the compatibility equation $\phi_{xt} = \phi_{tx}$.

We consider the Eq. (7), without loss of generality we take $d_n = 1$, so that, we may write

$$u(x,t) = a_n(x,t) + \theta_{n-1}(\phi), \qquad (34)$$
$$\theta_{n-1}(\phi) := \frac{P_{n-1}(\phi)}{Q_n(\phi)} = (\sum_{i=0}^{n-1} r_i(x,t)\phi^i)/(\phi^n + \sum_{i=0}^{n-1} d_i(x,t)\phi^i).$$

Lemma 3.1. *a_n satisfies KdV equation (10) if and only if $\theta_{n-1,n}$ verify the partial differential equation*

$$(\theta_{n-1})_t + f(x,t)(\theta_{n-1})_{xxx} + g(x,t)(\theta_{n-1})(\theta_{n-1})_x + (a_n\theta_{n-1})_x = 0. \qquad (35)$$

It is worth noticing that for $n \neq 1$ we have a generalized B$\ddot{a}$cklund transformation, but for $n = 1$ it is auto-B$\ddot{a}$cklund transformation.

Theorem 3.2. *When* $n=1$, *a rational solution to (10) exists if and only if* $f(x,t)=cx^2 g(x,t)$, $g(x,t)=xg(t)$ *where* c *is a constant.*

Proof. If $f(x,t)=cx^2 g(x,t)$, $g(x,t)=xg(t)$, we use $\tau=\int g(t)dt$, $z=\ln x$, and $u=u$, thus (10) becomes a KdV with constant coefficients. It has a soliton solution in the variable $\tilde{z}=z-k\tau$, k is a constant. This solution is rational in the exponential function that satisfies the auxiliary equation (33) when $c_{0,1}$ and $c_{1,1}$ are constants. Now, we prove that if a rational solution exists then $f(x,t)=cx^2 g(x,t)$ by the converse statement. We assume that there exists $f(x,t)\neq cx^2 g(x,t)$ and a solution exists when $n=1$, as

$$u(x,t)=a_1(x,t)+\theta_0(\phi),\theta_0(\phi)=\frac{r_0(x,t)}{\phi+d_0(x,t)}. \quad (36)$$

For Simplicity we assume that $f(x,t)=f_0(t)$, $g(x,t)=g_0(t)$, and $f_0(t)=s(t)g_0(t)$, so that $a_1(x,t)$ is given by

$$a_1(x,t)=\frac{-12s(t)}{x^2}+\frac{x}{s(t)},s(t)=A_0+\int g_0(t)dt, \quad (37)$$

where A_0 is a constant.

In this case, from the principle equations, namely those arising from substituting into (35), we get $b_1(x,t),b_0(x,t)$ and two other equations; namely

$$A(x,t)(A_0(x,t)+d_0(x,t)B(x,t))=0,A(x,t)(B_0(x,t)+B(x,t))=0, \quad (38)$$

$$A(x,t)=c_0(x,t)-d_0(x,t)c_1(x,t)+d_{0x}(x,t)=A_0(x,t)-d_0(x,t)B(x,t),$$

where $A_0(x,t)$, $B_0(x,t)$, and $B(x,t)$ are functions in $d_0,r_0,c_0,c_1,d_{0x},d_{0xx},\ldots$. The equation (38) has the unique solution $A(x,t)=0$. So that the auxiliary equations (34) has the solution

$$\phi(x,t)+d_0(x,t)=c_{1,1}e^{\int c_1(x,t)dx}. \quad (39)$$

Where, c_{11} is a constant.

Thus, no rational solution exists $(Cf.(36)_2)$ unless $f(t)=cg(t)$.

For $n\geq 2$, computations are too lengthy and they will be considered in a future work.

4 Conclusions

In this paper, we suggested an extended unified method for finding exact solutions to evolution equations with variable coefficients. A wide class of exact solutions to KdV equation with Space-time dependent coefficients is obtained. The method and the solutions that we obtained here are completely new and we can use this method to find exact solutions of coupled evolution equations. But in this case we think that parallel computations should be used.

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