Application of Collocated Legendre Expansions on Some Ibvps

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Abstract: The aim of this paper is numerical solution of some IBVPs with various boundary conditions. Here, we approximated some second order IBVPs using Legendre special function expansion. Then, we use Legendre polynomials on the interval [-1, 1] as the basis functions (i. e. shape or approximation functions) for collocation method. We also evaluate conditions that can give better approximations, then present some examples of different boundary conditions and finally our approximate solutions are compared with its analytic solution.
[Ghorbani Mehrzad, Nafisinia Yunes,Ghorbani jafarbeigloo Roya. **Application of collocated Legendre expansions on some IBVPs.** *Life Sci J* 2013;10(1):1004-1009] (ISSN:1097-8135). http://www.lifesciencesite.com. 156

Keywords: collocation method, second order IBVP, Legendre polynomials, nodal distribution.

0 – Introduction:

There are many phenomena in science and engineering that can be formed as differential equations together with boundary conditions (BCs) and for most of them their analytic solutions can not found. So, they can numerically be approximated by numerical methods. The finite difference methods (FDMs) are one of the ancient and simplest approximation methods that are applicable on most of the PDEs. The two main disadvantages of the FDMs are: its nodal or point-wise approximation (i. e., finding approximations on discrete points) and the other is restriction of the method on problems with rectangular, square and circular domains [8].
The finite element methods (FEMs) [18], finite volume methods (FVMs) [11], multi-grid methods (MGMs) [19] and mesh-free methods [10] are the other approximation methods [20].

The weighted residual methods (WRMs) that belong to FEMs have not the two mentioned FDM disadvantages. The collocation method is one of the WRMs in which its weight is Dirac delta distribution, and here, we employed and used of it and is usually used for approximating ODEs, PDEs and integral equations (IEs). As stated, unlike the FDMs, the WRMs give a continuous function solution in spite of solutions on discrete points [6].

In 1782, Legendre polynomials introduced [23] as the coefficients of Newton potential problem. In 1984, Bellen used Legendre orthogonal polynomial for one-step collocation method and for solution of delay differential equations (DDEs) [1].

In 1992, Mullenheim gave little attention on solving 2nd order nonlinear BVPs with Dirichlet, Neumann and Robin BCs [9], then in 2007, BVPs together with Neumann BCs with Splines are employed [14].

In [2], a typical linear mode example using collocation method examined.

In 2011, ShafigulIslam et. al [15], studied BVPs with different BCs (Dirichlet, Neumann and Robin) using Galerkin method. Also, the BVPs in linear mode using the FDMs investigated in [22] and in refs. [13] and [23] the Spline polynomials which are smooth up to order 2 and can be linearly expanded by the Legendre polynomials were used.

The Legendre polynomials can be used for approximating ODEs and also this polynomial can be applied on polar form of the Laplace equation.

The subjects of this paper are as follows: Section 1 deals with IBVP formulations according to the 2nd order Legendre polynomials and in section 2, the collocation method for approximating IBVP is explained. Some numerical experiments are presented in Section 3 together with Legendre polynomials and their errors.

In the final Section, some results, concluding remarks, research ideas and propositions will be given.

1 -Legendre polynomials:

The orthogonal polynomials play the most important role in the spectral and generalized fourier methods. So, it is necessary to have a view on their relevant properties. Our starting point is being generated orthogonal polynomials by a recurrence relations, which leads to some useful formulas such as Christoffel-Darboux and Rodrigues formula [7].
The Legendre polynomials are one of the most famous orthogonal polynomials. The Legendre functions $p_n(x)$, are solutions of the following Legendre ODE for different integer n:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}p_n(x)\right]+n(n+1)p_n(x) = 0. \qquad (1-1)$$

The general solution of the Eq. (1-1) is (see [7] and [23]):

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]. \qquad (1-2)$$

After expanding the expression of the Eq. (1-2), the Legendre solutions will be:

$$p_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(x-1)^{n-k}(x+1)^k = \sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{k}\left(\frac{1-x}{2}\right)^k = 2^n\sum_{k=0}^{n}x^k\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}. \qquad (1-3)$$

The Legendre polynomials are defined in the interval $[-1,1]$ and their weights are $w(x)\equiv 1$. They are also orthogonal and continuous [7], so, they are good candidate for base functions of the collocation method that we applied on our problems.

In general, a two point 2nd order differential equation (linear or non linear) is in the following form [24]:

$$u''(x) = f(x,u,u'), \qquad a \le x \le b, \qquad (1-4a)$$

$$A_1u(a) - A_2u'(a) = \alpha, \qquad |A_1| + |A_2| \ne 0, \quad (1-4b)$$

$$B_1u(b) + B_2u'^{(b)} = \beta, \qquad |B_1| + |B_2| \ne 0, \quad (1-4c)$$

and its linear form is as follows:

$$u''(x) - p(x)\,u'(x) - q(x)\,u(x) = r(x), \qquad a \le x \le b, \qquad (1-5a)$$

$$A_1u(a) - A_2u'(a) = \alpha, \qquad |A_1| + |A_2| \ne 0, \qquad (1-5b)$$

$$B_1u(b) + B_2u'(b) = \beta, \qquad |B_1| + |B_2| \ne 0, \qquad (1-5c)$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions, and the parameters A_1, A_2, B_1, B_2 are constants. The two problems (1-4) and (1-5) are called IBVPs (mixed BCs) and according to Picard-Lindelof theorem, if $u(x)$ be continuous and Lipschitz, these equations will have a unique solution [5].

The problem (1-5) in operator form is as follows:

Let, $X = \{u\epsilon C^2[a,b]: A_1u(a) - A_2u'(a) = \alpha \;\&\; B_1u(b) + B_2u'(b) = \beta\}$, $L: X \to C[a,b]$ and $L = \frac{d^2}{dx^2} - p(x)\frac{d}{dx} - q(x)$, thus the problem (1-5a) is equivalent to:

$$L(u(x)) = r(x). \qquad (1-5d)$$

Let the Legendre polynomials are given on the interval $[-1,1]$. By substitutions x and $\xi = \frac{(b-a)(x+1)}{2} + a$, the interval $[-1,1]$ can be mapped into the interval $[a,b]$, and by using this change of variable, the Eq. (1-5) will be changed as follows:

$$u''(x) - \bar{p}(x)u'(x) - \bar{q}(x)u(x) = \bar{r}(x), \qquad -1 \le x \le 1, \qquad (1-6a)$$

$$A_1u(-1) - \frac{2A_2}{b-a}u'(-1) = \alpha, \qquad (1-6b)$$

$$B_1u(1) + \frac{2B_2}{b-a}u'(1) = \beta, \qquad (1-6c)$$

and the coefficient functions will be changed into the following forms:

$$\bar{p}(x) = \frac{b-a}{2}p\left(\frac{(b-a)(x+1)}{2} + a\right),$$

$$\bar{q}(x) = \frac{(b-a)^2}{4}q\left(\frac{(b-a)(x+1)}{2} + a\right),$$

$$\bar{r}(x) = \frac{(b-a)^2}{4}r\left(\frac{(b-a)(x+1)}{2} + a\right).$$

2 - The collocation method and its application:

The collocation idea is:

1. the selection a finite-dimensional space of candidate base functions (usually, a complete polynomials up to a certain degree).
2. the selection of a finite number of points distributed irregularly in the domain area (called collocation points). The experiments show that points must have more compact in high gradient part of the domain for getting better approximation.
3. and then selecting an approximation expansion function which satisfies the given equation at all of the collocation points.

As considered in Section 1, the operator L is linear and X the solution space of the problem (1-5) is a real vector space, consequently for finding an approximate solution of the Eq. (1-5d), and the linear and transformed IBVP (1-6), in a $n+2$ dimensional subspace $Y \subset X$, and by applying the collocation method, we do as follows [17]:

Let,

$$\tilde{u}_n(x) = \sum_{j=0}^{n+1} c_j\,\varphi_j(x), \qquad (2-1)$$

be the selected approximation, such that in which c_j, $j = 0,1,\dots,n+1$ are unknown constants, and $\varphi_j(x)$, $j = 0,1,\dots,n+1$ are elements of approximating $n+2$ dimensional subspace Y of X. The experiments showed better results can be obtained by relating the collocation points to the structure of classical orthogonal polynomials such as Chebyshev, Hermite, Bessel, Lagurre and Legendre polynomials [7].

The selected collocation points are: $t_0 = -1$, $t_{n+1} = 1$, and for $i = 1,\dots,n$, t_i is chosen as the roots of the base functions (here, the Legendre functions) of order n and for finding residual and spectral approximation, we must express the derivatives of $\tilde{u}_n(x)$ in terms of u at the collocation points t_i. Thus, the derivative relations of the Eq. (2-1) are:

$$\tilde{u}'_n(x) = \sum_{j=0}^{n+1} c_j\, \varphi'_j(x), \quad \tilde{u}''_n(x) = \sum_{j=0}^{n+1} c_j\, \varphi''_j(x),$$

and then, the application of the collocation method on the Eq. (1-6) and the collocation points t_i , i = $0, 1, \dots, n+1$ will be in the following forms:

$$\tilde{u}''_n(t_i) - \bar{p}(t_i)\, \tilde{u}'_n(t_i) - \bar{q}(t_i)\, \tilde{u}(t_i) - \bar{r}(t_i) = 0, \quad -1 \le t_i \le 1, \quad i = 1, 2, \dots, n, \quad (2-2a)$$

$$A_1 \tilde{u}_n(t_0) - \frac{2A_2}{b-a}\tilde{u}'_n(t_0) - \alpha = 0, \qquad (2-2b)$$

$$B_1 \tilde{u}_n(t_{n+1}) + \frac{2B_2}{b-a}\tilde{u}'_n(t_{n+1}) - \beta = 0, \qquad (2-2c)$$

This means that residual of the equation must be zero in all of the collocation points.

The Eq. (2-2) tends to a diagonally dominant linear system [3]. Therefore, this system will have a unique solution, $det[\varphi_j(x_i)] \neq 0$ for $i, j = 0, 1, 2, \dots, n+1$. Consequently, the base function system $\{\varphi_0(x), \varphi_1(x), \dots, \varphi_{n+1}(x)\}$ is linearly independent over [-1, 1]. By solving this system using direct (such as: L-U) or iterative methods (such as: Jacobi, Gauss-Seidel or SOR) [3], the approximate solution $\tilde{u}_n(x)$ will be obtained [24]. As stated before this, the suitable selection of the collocation points for obtaining an acceptable precision approximate solution $\tilde{u}_n(x)$ is very important. Also, by finding high precision roots of the Legendre polynomials, the accuracy of the approximate solution will be raised. The coefficient matrix of the collocation system is always full and dense with a condition number behaving like O(N^{2m}) (m is the order of the differential equation) [12].

3 - Numerical experiments:

In this section, we try to approximate some IBVP together with various BCs. In examples (3-1) and (3-2), the roots of the Legendre polynomial $L_8(x)$ will be used as the collocating points of approximation and in example (3-3), the roots of the Legendre polynomial $L_{10}(x)$ will be used.

We used the Mathematica 8.01 for programming and numerical results were shown by a table and a graph. Let, $u(x)$ denote analytical solution, $\tilde{u}_n(x)$ its related approximation, and $\|u(x) - \tilde{u}_n(x)\|$ be approximation error.

Example 3-1: In this example, we consider the following BVP with Dirichlet BCs [16].

$$\frac{d^2u}{dx^2} + u = x^2 e^{-x}, \qquad 0 \le x \le 10,$$

$$u(0) = u(10) = 0.$$

Its equivalent and transformed equation on the normalized interval [-1, 1] is:

$$\frac{d^2u}{dx^2} + 25u = 25(x+1)^2 e^{-5(x+1)}, \qquad -1 \le x \le 1,$$

$$u(-1) = u(1) = 0,$$

and the analytical solution of the problem is:

$$u(x) = \frac{1}{50} e^{-5(2+x)} csc10\big[e^5(6+5x)^2 sin10 - e^{5(2+x)} sin(5-5x) - 121 e^{5x} sin\big(5(1+x)\big)\big].$$

Using the expansion (1-2), and applying the collocation method, its approximate solution will be found and its result comes in Table 1 and Figure 1 as is shown.

Table 1 shows the approximate solution and the absolute error of the example (3-1)

The graph that compares the analytical and approximate solution of the example (3-1) is as follows:

Figure 1 shows a comparison between analytical and approximate solution (dashed) of the example (3-1).

Example 3-2: Here, we consider a problem with Neumann BCs [14].

$$\frac{d^2u}{dx^2} + u = -1, \qquad 0 \le x \le 1,$$

$$u'(0) = \frac{1-cos1}{sin1}, \quad u'(1) = -u'(0).$$

The transformed equation into the interval [-1, 1] will be as follows:

$$\frac{d^2u}{dx^2} + \frac{1}{4}u = -\frac{1}{4}, \quad -1 \le x \le 1,$$

$$u'(0) = \frac{1-cos1}{2sin1}, \quad u'(1) = -u'(0),$$

and its analytical solution is:

$$\mathrm{u}(x) = -1 + \mathrm{Cos}\left[\frac{x}{2}\right] \mathrm{Sec}\left[\frac{1}{2}\right].$$

The result of the numerical solution of this problem is given in Table 2 (Table 2: The approximation solution and its absolute error of the example (3-2)). The graph for comparing analytical and approximate solution of the example (3-2) is showed as follows:

Figure 2: Graph of comparing between analytical and approximate (dashed) solution of the example (3-2).

Example 3-3: Here we applied the method, to the following problem with mixed or Robin BCs [19].

$$-\frac{d^2u}{dx^2} + u = 2\,cosx, \qquad \frac{\pi}{2} \le x \le \pi,$$

$$u'\left(\frac{\pi}{2}\right) + 3u\left(\frac{\pi}{2}\right) = -1,$$

$$u'(\pi) + 4u(\pi) = -4.$$

The transformed form of the problem on the interval [-1, 1] is as follows:

$$\frac{d^2u}{dx^2} - \frac{\pi^2}{16}u = -\frac{\pi^2}{8} cos\left(\frac{\pi}{4}x + \frac{3\pi}{4}\right), \qquad -1 \le x \le 1,$$

$3u(-1)+\frac{4}{\pi}u'(-1)=-1,$

$4u(1)+\frac{4}{\pi}u'(1)=-4,$

and its analytical solution is:

$$u(x)=-\frac{\cos\left(\frac{\pi}{4}x\right)+\sin\left(\frac{\pi}{4}x\right)}{\sqrt{2}}.$$

The numerical result of the method applied on this problem is specified in Table 3:

Table 3 shows the approximate solution and its absolute error of the example (3-3).

The geometrical comparison between related analytical and approximate solution is as follows:

Figure 3: The comparison between analytical and approximate (dashed) solution of the example (3-3).

The approximate solutions, shown in the Tables 1 and 2 are obtained based on the use of $L_8(x)$ and in Table 3 obtained based on the use of $L_{10}(x)$ and the accuracy is observed nearly 3, 4 decimal places, respectively.

The use of uniformly distributed collocation points, gives more error than distribution of nodes by Legendre polynomial roots.

In our final experiment, we re-approximated the example (3-3) by uniformly spaced points ($h=0.2$, $h=\frac{b-a}{n+1}$). The error in uniformly spaced data points at: -0.9 and 0.9 is: 2.539×10^{-2} and 6.214×10^{-3} respectively, but as shown in the example (3-3) are: 4.727×10^{-3} and 3.405×10^{-4}.

4 – Conclusion:

In this paper, we presented a numerical method in which the Legendre polynomials as the approximation functions and the collocation method was used, and also other special functions or orthogonal systems can be used. The more number of base functions tend into the higher accuracy approximation $\tilde{u}_n(x)$. The choice of the collocation points (situation and the numbers) is an important and influential factor on the final result and here, we applied the roots of the orthogonal Legendre polynomials as the collocation points and found an acceptable approximation. Instead of the Legendre functions, one can employ other functions such as special functions and also orthogonal wavelet functions [4]. This paper was limited on one-dimensional problems. Therefore it can be extended to higher dimensional problems. Instead of the collocation method, the other weighted residual methods such as: Galerkin, Petrov-Galerkin, least square and sub-domain methods can be applied. The use of the full matrix systems have computationally high prices, therefore by selecting locally compact support base functions, one can tend this method to sparse and lower price matrices.

Table 1: The approximate solution and the absolute error of the example (3-1)

x	$\tilde{u}_n(x)$	$u(x)$	$\lVert u(x)-\tilde{u}_n(x)\rVert$
-1.0	0.002943	0.000000	2.943×10^{-3}
-0.8	0.038872	0.044751	5.878×10^{-3}
-0.6	0.061156	0.060916	2.401×10^{-4}
-0.4	0.044200	0.040113	4.086×10^{-3}
-0.2	0.006194	-0.001267	7.462×10^{-3}
0.0	-0.022633	-0.030595	7.961×10^{-3}
0.2	-0.022922	-0.025449	2.527×10^{-3}
0.4	0.002108	0.006487	4.379×10^{-3}
0.6	0.028305	0.034172	5.866×10^{-3}
0.8	0.027686	0.031265	3.578×10^{-3}
1.0	0.002943	0.000000	2.943×10^{-3}

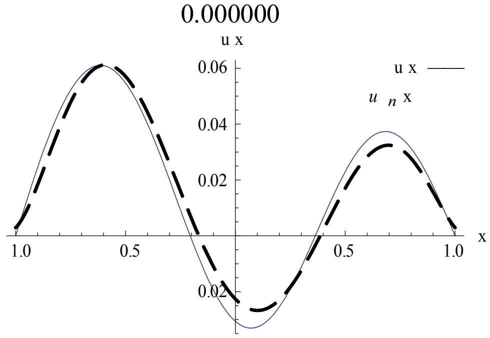

Figure 1: A comparison between analytical and approximate solution (dashed) of the example(3-1).

Table 2: The approximation solution and its absolute error of the example (3-2)

x	$\tilde{u}_n(x)$	$u(x)$	$\lVert u(x)-\tilde{u}_n(x)\rVert$
-1.0	-0.006385	0.000000	6.385×10^{-3}
-0.8	0.043579	0.049543	5.964×10^{-3}
-0.6	0.083858	0.088600	4.741×10^{-3}
-0.4	0.113350	0.116780	3.429×10^{-3}
-0.2	0.131700	0.133801	2.101×10^{-3}
0.0	0.138743	0.139494	7.508×10^{-4}
0.2	0.134390	0.133801	5.892×10^{-4}
0.4	0.118671	0.116780	1.890×10^{-3}
0.6	0.091734	0.088600	3.143×10^{-3}
0.8	0.053833	0.049543	4.290×10^{-3}
1.0	0.005151	0.000000	5.151×10^{-3}

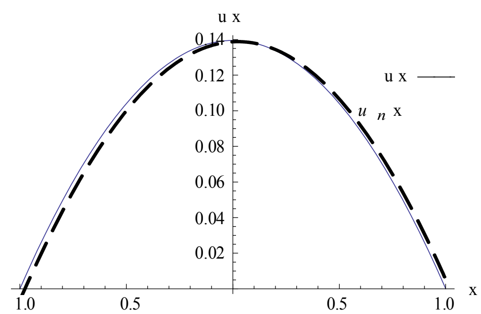

Figure 2: Graph of comparing between analytical and approximate (dashed) solution of the example (3-2).

Table 3: The approximate solution and its absolute error of the example (3-3).

x	$\tilde{u}_n(x)$	$u(x)$	$\lVert u(x)-\tilde{u}_n(x)\rVert$
-1.0	-0.004880	0.000000	4.880×10^{-3}
-0.8	-0.151673	-0.156434	4.761×10^{-3}
-0.6	-0.304992	-0.309017	4.025×10^{-3}
-0.4	-0.450677	-0.453990	3.313×10^{-3}
-0.2	-0.585051	-0.587785	2.734×10^{-3}
0.0	-0.704937	-0.707107	2.169×10^{-3}
0.2	-0.807283	-0.809017	1.734×10^{-3}
0.4	-0.889723	-0.891007	1.283×10^{-3}
0.6	-0.950222	-0.951057	8.349×10^{-4}
0.8	-0.987386	-0.987688	3.023×10^{-4}
1.0	-1.000820	-1.000000	8.200×10^{-4}

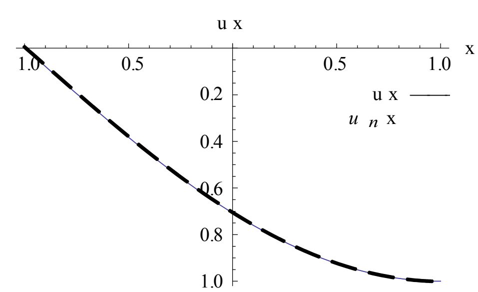

Figure 3: The comparison between analytical and approximate (dashed) solution of the example (3-3).

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12/25/2012