Space Adaptive Numerical Scheme to Solve Black-Scholes Equation.

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Abstract: A grid adaptive finite difference technique is developed to evaluate digital call option for one asset using Black-Scholes equation. The grid is refined near the exercise price and a coarse grid is generated otherwise. To cope with these uneven space steps, an innovative numerical scheme is developed. The numerical experiments show that the adaptive finite difference method is much more efficient than the method with uniform spacing. An Implicit and Explicit grid adaptive finite difference techniques are established to work with non-uniform grids

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1. Introduction

In the present world of finance, there are many types of financial instruments (Duffy, 2006) which go by the name of Options. Options are traded on all of the world's major exchanges. Binary options or digital options (Khaliq, et al, 2007) are not only very popular in the over-the-counter(OTC) markets but also important tools for designing more complex financial derivatives (Wilmott and Howison, 1996). For example, holding the simplest cash-or-nothing call option pays a predefined cash amount at the expiry date if the option is in-the money. Therefore, at the strike price, the payoff has a discontinuity. In this work, we will focus on digital call options for one asset.

Black-Scholes and Merton (Black and Scholes, 1973) derived a celebrated partial differential equation. The Black-Scholes model is the best way to calculate the price of an option (Cox et al, 1979). In this article numerical methods (Smith, 1985) will be used to solve the finite difference equation (Courtadon, 1982) of Black-Scholes. Even though the solution to the Black-Scholes equation is smooth, the final condition has discontinuity which produces oscillation in the numerical solution. In order to cure this oscillation from the initial discontinuities, there have been studied different numerical methods (Dura and Mosneagu, 2010. Zhu et al, 1988) in many application areas. Finite difference methods (Khaliq et al, 2008. Wade et al, 2007) with variable space-steps are proposed in order to valuate binary options.

The purpose of this paper is to develop efficient and accurate numerical methods to price options (Zhongdi and Anbo, 2009) with payoff containing discontinuities. For standard binary options, the discontinuity lies only in the initial condition, therefore we need to use small space-steps initially then use bigger space-steps to keep the efficiency. In proposed study, we focus on adaptivity (Hongjoong, 2011) for space-steps in order to see effects of variable space-steps. In this study, several numerical tests show that the adaptive finite difference methods approximate the solution more efficiently than uniform finite difference methods.

2. Proposed Discretization

Let S(t) be the price of the underlying asset at time t ($0 \le t \le T$) with a given expiry date T, constant interest rate r >0 and a constant volatility $\sigma > 0$. The value, $V(S,t)$ of binary options under classical Black-Scholes model can be computed by solving the following one asset partial differential equation,

$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rV = 0. \qquad (2.1)$$

Digital call (cash-or-nothing) options for one asset pay a cash amount A at expiration if the option is in-the-money that is,

$$\Lambda(S) = \begin{cases} A & \text{if } S \ge E \\ 0 & \text{otherwise} \end{cases},$$

where E >0 is a predefined exercise price and $\Lambda(S)$, is the payoff function at expiry date T.

The interval $[0,T]$ is divided into M equally sized subintervals of length Δt. The price of underlying asset will take the values in the unbounded interval $[0,\infty)$. However, an artificial limit $S_{\max}$ is introduced. The size of $S_{\max}$ requires experimentations; but normally $S_{\max}$ is taken around three to four times the exercise price E. The interval $[0,S_{\max}]$ is divided into N subintervals of length ΔS_i. The asset price at an arbitrary point n will be

$\sum_{i=0}^{n} \Delta S_i = \Delta S_0 + \Delta S_1 + \Delta S_2 + \cdots \Delta S_{N-1} + \Delta S_n$ Let us assign a variable α_n to this summation, then $\alpha_n = \sum_{i=0}^{n} \Delta S_i$.

Using this nomenclature, we can say that $\alpha_N = \sum_{n=0}^{N} \Delta S_n = S_{\max}$. where ΔS_i are the non-uniform space-steps. Hence, the space $\left[0, S_{\max}\right] \times [0, T]$ is approximated by a grid $(\alpha_n, m\Delta t) \varepsilon [0, \alpha_N] \times [0, M\Delta t]$,

where $n = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, M$. For uniform spacing $\alpha_n = n\Delta S_n$. Let V_n^m denote the numerical approximation of $V(\alpha_n, m\Delta t)$. The time derivative V_t can be approximated as,

$$V_t(\alpha_n, m\Delta t) = \frac{V_n^{m+1} - V_n^m}{\Delta t} + o(\Delta t). \qquad (2.2)$$

The first spatial derivative V_S is given as

$$V_S(\alpha_n, m\Delta t) = \frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{2\Delta S_n} + o(\Delta S_n)^2. \qquad (2.3)$$

The second spatial derivative V_{SS} is given by

$$V_{SS}(\alpha_n, m\Delta t) = \frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\Delta S_n)^2} + o(\Delta S_n)^2, \qquad (2.4)$$

where a space-step size $\Delta S_n = S_{n+1} - S_n$ is assumed uniform but ΔS_n can be different in our case. The above equation (2.3) and equation (2.4) can be easily modified for variable spacing as follows,

$$V_S(\alpha_n, m\Delta t) \approx \frac{V_{n+1}^m - V_{n-1}^m}{\Delta S_n + \Delta S_{n-1}}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{\Delta S_{n-1}(V_{n+1}^m - V_n^m) - \Delta S_n(V_n^m - V_{n-1}^m)}{(\Delta S_n)^2 \times \Delta S_{n-1}}$$

In digital option the discontinuity appears at exercise price. In the proposed procedure, dense grid is generated in the vicinity of the exercise price and coarse grid is generated else where. Hence, the whole space can be divided into three patches of points as shown in figure. The patch I and patch III has coarse grids while in patch II the dense grid is generated. The grid in each patch is uniform therefore, the order of the error in each patch is the same as for uniform grid i.e. $o(\Delta S_n)^2$. But at the two intersection points the order of the numerical scheme is reduced.

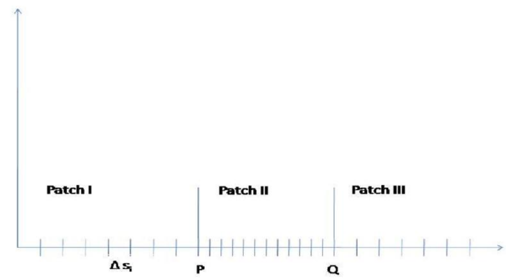

Figure 1. Patch II is in the vicinity of discontinuity

2.1 Adaptive Explicit Finite Difference Scheme

Following is the discretized Equation (2.1) for non uniform grid :

$$\frac{V_n^{m+1} - V_n^m}{\Delta t} - r \sum_{i=0}^{n} \Delta S_i \left(\frac{V_{n+1}^m - V_{n-1}^m}{\Delta S_n + \Delta S_{n-1}}\right)$$

$$-\frac{1}{2}\sigma^2\left(\sum_{i=0}^{n} \Delta S_i\right)^2 \left\{\frac{\Delta S_{n-1}(V_{n+1}^m - V_n^m) - \Delta S_n(V_n^m - V_{n-1}^m)}{(\Delta S_n)^2 \times \Delta S_{n-1}}\right\} + rV_n^m = 0$$

Simplifying and re-arranging, the above equation takes the form:

$$V_n^{m+1} = \Delta t\left(\frac{\sigma^2 \alpha_n^2}{2\Delta S_n \times \Delta S_{n-1}} - \frac{r\alpha_n}{\Delta S_n + \Delta S_{n-1}}\right) V_{n-1}^m$$

$$+\left\{1 - \Delta t \frac{\sigma^2 \alpha_n^2}{2\Delta S_n}\left(\frac{1}{\Delta S_n} + \frac{1}{\Delta S_{n-1}}\right) - r\Delta t\right\} V_n^m$$

$$+\Delta t\left(\frac{\sigma^2 \alpha_n^2}{2(\Delta S_n)^2} + \frac{r\alpha_n}{\Delta S_n + \Delta S_{n-1}}\right) V_{n+1}^m.$$

The term V_n^{m+1} at $m+1$ in explicit form is evaluated using the terms $V_{n-1}^m, V_n^m, V_{n+1}^m$, . Let

$$A = \begin{bmatrix} d_1 & u_2 & 0 & \cdots & \cdots & 0 \\ l_1 & d_2 & u_3 & 0 & & \vdots \\ 0 & l_2 & d_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & u_{N-1} \\ 0 & \cdots & \cdots & 0 & l_{N-2} & d_{N-1} \end{bmatrix},$$

where $A \in R^{(N-1)\times(N-1)}$,

$$V^{m+1} = \begin{bmatrix} V_1^{m+1} \\ V_2^{m+1} \\ \vdots \\ V_{N-1}^{m+1} \end{bmatrix}, V^m = \begin{bmatrix} V_1^m \\ V_2^m \\ \vdots \\ V_{N-1}^m \end{bmatrix}, Z^m = \begin{bmatrix} l_0 V_0^m \\ 0 \\ \vdots \\ 0 \\ u_N V_N^m \end{bmatrix},$$

where $V^{m+1}, V^m, Z^m \varepsilon R^{N-1}$

$$d_n = 1 - \frac{\Delta t \sigma^2 \alpha_n^2}{2\Delta S_n}\left(\frac{1}{\Delta S_n} + \frac{1}{\Delta S_{n-1}}\right) - r\Delta t,$$

n=1, . . . , N-1

$$u_n = \Delta t\left(\frac{\sigma^2 \alpha_{n-1}^2}{2(\Delta S_{n-1})^2} + \frac{r\alpha_{n-1}}{\Delta S_{n-2} + \Delta S_{n-1}}\right),$$

n=2, . . . , N,

$$l_n = \Delta t\left(\frac{\sigma^2 \alpha_{n+1}^2}{2\Delta S_n \times \Delta S_{n+1}} - \frac{r\alpha_{n+1}}{\Delta S_n + \Delta S_{n+1}}\right),$$

n=0, . . . , N-2.

The equation (2.5) can then be written in matrix form as :

$$V^{m+1} = AV^m + Z^m.$$

We observe that at every time-step $m+1$, the approximate solution can be obtained from the above matrix equation. The values V_n^0, V_0^m, V_N^m with $n = 0,\ldots,N$ and $m = 0,\ldots,M$ are known from initial and boundary conditions. By taking L_2- norm, following condition of stability can be deduced,

$$0 < \Delta t < \frac{1}{\sigma^2 \alpha_{N-2}^2 \beta + \frac{r}{2}}$$

where $\beta = \dfrac{\Delta S_{n-3} + \Delta S_{n-2}}{2(\Delta S_{n-2})^2 \times \Delta S_{n-3}}$

2.2 Adaptive Backward-Euler Finite Difference Scheme

In this method, we use forward difference for V first time derivative, central difference for first S derivative and for second S derivative, we first use forward difference and then backward difference:

$$\frac{\partial V}{\partial t}(\alpha_n, (m+1)\Delta t) \approx \frac{V_n^{m+1} - V_n^m}{\Delta t},$$

$$\frac{\partial V}{\partial S}(\alpha_n, (m+1)\Delta t) \approx \frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{\Delta S_n + \Delta S_{n-1}},$$

$$\frac{\partial^2 V}{\partial S^2}(\alpha_n, (m+1)\Delta t)$$

$$\approx \frac{\Delta S_{n-1}(V_{n+1}^{m+1} - V_n^{m+1}) - \Delta S_n(V_n^{m+1} - V_{n-1}^{m+1})}{(\Delta S_n)^2 \times \Delta S_{n-1}}.$$

Using the above substitutions, equation (2.1) takes the form :

$$\frac{V_n^{m+1} - V_n^m}{\Delta t} - r\sum_{i=0}^{n} \Delta S_i \left(\frac{V_{n+1}^m - V_{n-1}^m}{\Delta S_n + \Delta S_{n-1}}\right)$$

$$-\frac{1}{2}\sigma^2\left(\sum_{i=0}^{n} \Delta S_i\right)^2$$

$$\frac{\Delta S_{n-1}(V_{n+1}^{m+1} - V_n^{m+1}) - \Delta S_n(V_n^{m+1} - V_{n-1}^{m+1})}{(\Delta S_n)^2 \times \Delta S_n}$$

$$+rV_n^{m+1} = 0$$

After simplifying and re-arranging, the above equation takes the form :

$$\left(1 + \frac{\Delta t \sigma^2 \alpha_n^2}{(\Delta S_n)^2} + \frac{\Delta t \sigma^2 \alpha_n^2}{2\Delta S_n \times \Delta S_{n-1}} + r\Delta t\right) V_n^{m+1}$$

$$= V_n^m - \left(\frac{r\Delta t\, \alpha_n}{\Delta S_n + \Delta S_{n-1}} - \frac{\Delta t \sigma^2 \alpha_n^2}{2\Delta S_n \times \Delta S_{n-1}}\right) V_{n-1}^{m+1}$$

$$+ \left(\frac{\Delta t \sigma^2 \alpha_n^2}{2(\Delta S_n)^2} + \frac{r\Delta t\, \alpha_n}{\Delta S_n + \Delta S_{n-1}}\right) V_{n+1}^{m+1}.$$

This system of equations can be solved by Gauss-Seidel method. The values V_n^0, V_0^m, V_N^m with $n = 0,\ldots,N$ and $m = 0,\ldots,M$ are known from initial and boundary conditions.

3. Numerical Experiments

We demonstrate some numerical experiments for one asset for the digital call option. In digital call option, the payoff is acting as the initial condition and has a piecewise discontinuity at the strike price. For digital call option or binary option (Khaliq et al, 2007), the payoff is 0 before the strike price and A, after the strike price. It is also known as cash or nothing option. A digital option differ from European call option in that the payoff at expiry is

$$\begin{cases} A, \text{if } S(t) \ge E, \\ 0, \text{if } S(t) < E, \end{cases}$$

where $A > 0$, is fixed. Such type of options are usually traded between a bank and a customer.

We use the following parameters for the computation of the cash-or-nothing option for one asset: T=0.5, r=0.1, σ=0.4, E=15, S=40, A=1, with N=20, 40, 60, 80, 100, 120, 140, 160 grids in space, different schemes are applied for option valuation. Tables 1 and 2 show the option prices for an at-the-money(S=E) cash-or-nothing option from various schemes. In Tables 1 and 2, N and L show the number of points for uniform and variable space-stepping respectively, $C_{u,e}$ shows the option price for uniform spacing and $C_{a,e}$ shows the option price for adaptive spacing for

Explicit scheme. $C_{u,i}$ and $C_{a,i}$ are option prices for Implicit scheme. It can be observed that same option values are obtained by using less number of points in adaptive space-stepping as compared to uniform space-stepping and adaptive space-stepping converges more rapidly than uniform space-stepping

Table 1 Comparison between Explicit and adaptive explicit schemes option values

N	$C_{u,e}$	dif	L	$C_{a,e}$	dif
20	0.5777	---	25	0.5392	
40	0.5347	0.0430	50	0.5133	0.0259
60	0.5186	0.0261	75	0.5010	0.0082
80	0.5116	0.0070	100	0.5010	0.0041
100	0.5068	0.0048	125	0.4986	0.0024
120	0.5040	0.0028	150	0.4970	0.0016
140	0.5018	0.0022	175	0.4959	0.0011
160	0.5003	0.0015	200	0.4950	0.0009

Table 2 Comparison between Implicit and adaptive implicit schemes option values

N	$C_{u,i}$	dif	L	$C_{a,i}$	Dif
20	0.5781		25	0.5393	
40	0.5349	0.0432	50	0.5134	0.0259
60	0.5187	0.0162	75	0.5052	0.0082
80	0.5117	0.0070	100	0.5011	0.0041
100	0.5069	0.0048	125	0.4987	0.0024
120	0.5041	0.0028	150	0.4971	0.0016
140	0.5018	0.0023	175	0.4959	0.0012
160	0.5003	0.0015	200	0.4951	0.0008

Figure 2, depicts the grid for initial conditions for one asset. Here, we refined interval [E- ε , E+ ε] around the strike price. We choose ε(epsilon) as 5 and E=15. The grid is refined in the interval [E- ε , E+ ε] to cure oscillations caused by discontinuity.

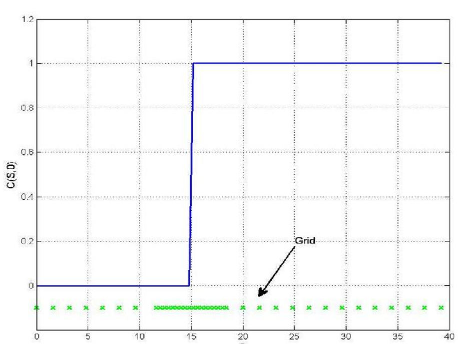

***Figure 2** Payoff function for one asset digital call option.*

Figure 3, represents graph of payoff function of digital call option for one asset. As is obvious from graph, in uniform coarse grid, in adaptive grid and in uniform dense grid, we have taken 50, 38 and 70 number of points respectively. It is clear that adaptive grid solution is very close to uniform dense grid solution but uniform coarse grid solution is away from uniform dense grid solution.

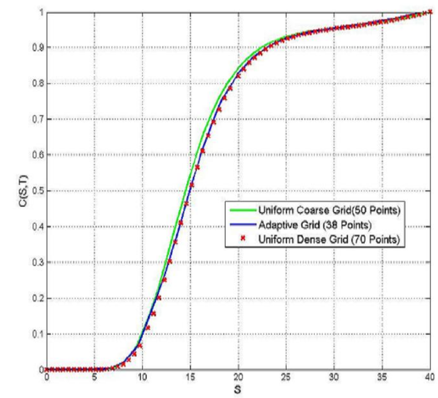

Figure 3 Simulation using adaptive explicit method

Figure 4, represents the Gamma plot for exact solution, uniform coarse/dense grid and adaptive grid solutions for one asset by explicit method. Here also, we see that adaptive grid plot matches with the plot of uniform dense grid and plot of exact solution.

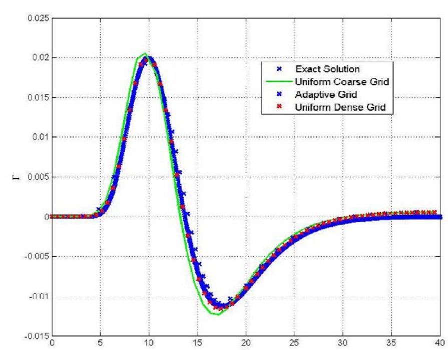

Figure 4 Gamma plot for T=0.5, r=0.1,σ =0.4, E=15, S=40, A=1

From figures, it is obvious that in adaptive space-stepping, we used less number of points but solution is nearly similar to that of exact solution but in uniform space-stepping, the solution does not match to exact solution when we use less number of points. Similar results can be obtained for space-stepping by Backward-Euler scheme. This shows that adaptive space-stepping is much better than uniform space-stepping.

4. Conclusions

We have developed an efficient finite difference numerical technique for one asset to cure oscillations in the solution. The computational domain is descretized embedding more points near the singularities and coarse grid otherwise. We have to modify the numerical scheme to deal with the uneven spacing of the points. The stability analysis of explicit scheme is also performed for one asset Black-Scholes equation. The results are presented for an adaptive explicit scheme, and adaptive implicit scheme. The oscillations at discontinuities are eliminated by using adaptive space-stepping. The adaptive space-stepping speeds up the solution convergence as compared to the uniform space-stepping. The adaptive finite difference scheme needs less points in its computation and hence is very efficient.

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