

## An Enhanced Solution of the Universal Lambert's Problem

Alshaery A. A.

Department of Mathematics, Sciences Faculty for Girls, King AbduAziz University, Jeddah, Saudi Arabia  
[aaalshaery@kau.edu.sa](mailto:aaalshaery@kau.edu.sa)

**Abstract:** In this paper, an iterative method of arbitrary order of convergence ( $p \geq 2$ ) is developed for solving the universal Lambert's problem using homotopy continuation technique. The method does not need any priori knowledge of the initial guess, a property which avoids the critical situation between divergent to very slow convergent solution, that may exist in the application of other numerical methods depending on initial guess. Computational algorithms and numerical applications will be applied for some orbits.

[Alshaery. A. A. **An Enhanced Solution of the Universal Lambert's Problem.** *Life Sci J* 2012;9(4):5226-5229] (ISSN: 1097-8135). <http://www.lifesciencesite.com>. 777

**Keywords:** Lambert's Problem; Boundary value problem; orbit determination; Homotopy continuation method.

### 1. Introduction

The basic problems of space dynamics are initial and boundary value problem. In the present paper we shall consider the second one which is known as Lambert's problem.

Lambert's problem was studied in **Escobal 1965, Herrick 1971 and Battin 1964**. Also, in **1969 Lancaster and Blanchard** and **Mansfield 1989** established a unified forms of Lambert's Problem. In **1990, Gooding** developed a procedure for the solution. Recently, an algorithm was developed for Batten's method of the universal Lambert's Problem (**Alshaery 2008**).

The independent variables used in Lambert's Problem satisfied transcendental equation, which is usually solved by iterative methods, which in turn need: (1) initial guess, (2) an iterative scheme. In fact, these two points are not separated from each other, but there is a full agreement that even accurate iterative schemes are extremely sensitive to the initial guess. Moreover, in many cases the initial guess may led to drastic situation between divergent and very slow convergent solutions (**Sharaf et al. 2007**). In the field of the numerical analysis, very powerful techniques have been devoted (**Allgower and George 1990**) to solve transcendental equations without any priori knowledge of initial guess these techniques are known as homotopy continuation methods, this technique has been used in initial value problem leading to encouraging results (**Alshaery 2003**).

In this paper, an iterative method of arbitrary order of convergence ( $p \geq 2$ ) is developed for solving the universal Lambert's problem using homotopy continuation technique. The method does not need any priori knowledge of the initial guess, a property which avoids the critical situation between divergent to very slow convergent solutions that may exist in the application of other numerical methods depending on initial guess. Computational algorithms

and numerical applications will be applied for some orbits.

### 2. Mathematical Modelling

#### 2.1 Basic equation of Lambert's problem

In Lambert's problem, two positions vectors  $r_1 \equiv (x_1, y_1, z_1)$  ;  $r_2 \equiv (x_2, y_2, z_2)$  and the time interval between them  $\Delta t$  which assumed positive are given and its required to find the orbit passing through the two points. Then the quantities of Lambert's problem are; the lengths of the position vectors  $|r_1|$  ,  $|r_2|$  and the chord  $c$  between the two points, where

$$\begin{aligned} r_1 &= |r_1| = (x_1^2 + y_1^2 + z_1^2)^{\frac{1}{2}}, \\ r_2 &= |r_2| = (x_2^2 + y_2^2 + z_2^2)^{\frac{1}{2}}, \\ c &= |r_2 - r_1| \end{aligned}$$

$= \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{\frac{1}{2}}$   
and the related quantities:

$$\lambda_1^2 = r_1 + r_2 + c; \quad \lambda_2^2 = r_1 + r_2 - c.$$

The basic equation of the universal Lambert's theorem could be written as (**Vallado and McClain 2007**).

$$L(z) = C_1 Q_1(z) + C_2 Q_2(z) - \frac{2}{3} C_3 z^{-\frac{3}{2}} - \Delta t = 0, \quad (1)$$

where  $z = \frac{1}{a}$  [a is the semi-major axis of the orbit connecting the two points] and  $Q$ 's are given in terms of the hypergeometric function as:

$$Q_i(z) = F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z\lambda_i^2}{4}\right); \quad i = 1, 2, \quad (2)$$

$$C_1 = \pm \frac{\lambda_1^3}{6K}, \quad C_2 = \pm \frac{\lambda_2^3}{6K}, \quad K = \sqrt{\mu}, \quad (3)$$

where  $\mu$  is the gravitational parameter and the upper or lower sign is chosen depending on whether the orbital segment does not (case 1) or does (case 2) respectively include the attracting focus, while

$$C_3 = \begin{cases} 0 & \text{for case1} \\ \frac{-3\pi}{K} & \text{for case2} \end{cases} \quad (4)$$

**2.2 One-point iteration formulae**

Let  $Y(x) = 0$  such that  $Y: R \rightarrow R$  smooth map and has a solution  $x = \xi$  (say). To construct iterative schemes for solving this equation, will state some basic definition:

1. The error in the  $k^{th}$  iterate is defined as  $\epsilon_k = \xi - x_k$ .
2. If the sequence  $\{x_k\}$  converges to  $x = \xi$ , then  $\lim_{k \rightarrow \infty} x_k = \xi$ . (5)

3. If there exists a real number  $p \geq 1$  such that  $\lim_{i \rightarrow \infty} \frac{|x_{i+1} - \xi|}{|x_i - \xi|^p} = \lim_{i \rightarrow \infty} \frac{|\epsilon_{i+1}|}{|\epsilon_i|^p} = L \neq 0$ . (6)

The iterative scheme is of order  $p$  at  $\xi$ . The constant  $L$  is called a asymptotic error constant. For  $p = 1$ , the convergence is linear; for  $p = 2$ , the convergence is quadratic;  $p = 3, 4, 5$  the convergence is cubic, quadratic and quintic, respectively.

4. We shall consider only stationary one-point iteration formulae which has the form

$$x_{i+1} = R(x_i), \quad i = 0, 1, \dots \quad (7)$$

5. The order of one point iteration formulae could be determine either from: (a) The Taylor series of the iteration function  $R(x_n)$  about  $\xi$  (Ralston and Rabinowitz 1978) or from, (b) The Taylor series of the function  $Y(x_{k+1})$  about  $x_k$  (Danby and Burkard 1983).

By the last approach (b), it is easy to form a class of iterative formulae containing members of all integral orders (Sharaf and Sharaf 1998) to solve  $Y(x) = 0$  as

$$x_{i+1} = x_i + \sigma_{i,m+2}; \quad i = 0, 1, 2, \dots \quad m = 0, 1, 2, \dots \quad (8)$$

where

$$\delta_{i,m+2} = \frac{-Y_i}{\sum_{j=1}^{m+1} (\delta_{i,m+1})^{j-1} Y_i^{(j)} / j!}; \quad \delta_{i,1} = 1; \quad \forall i \geq 0, \quad (9)$$

$$Y_i^{(j)} = \frac{d^j Y(x)}{dx^j} \Big|_{x=x_i}; \quad Y_i \equiv Y_i^{(0)}. \quad (10)$$

The convergence order is  $m + 2$ , and given as

$$\epsilon_{i+1} = -\frac{1}{(m+2)!} \frac{Y(\xi)^{(m+2)}}{Y_i^{(1)}(\xi_1)} \epsilon_i^{m+2}, \quad (11)$$

where  $\xi$  between  $x_{i+1}$  and  $x_i$  and  $\xi_1$  between  $x_{i+1}$  and  $\xi$ .

**2.3 Homotopy continuation method for solving  $Y(x) = 0$**

Suppose one wishes to obtain a solution of a single non-linear equation in one variable  $x$  (say)

$$Y(x) = 0, \quad (12)$$

where  $Y: R \rightarrow R$  is a mapping which, for our application assumed to be smooth, that is, a map has as many continuous derivatives as requires. Let us consider the situation in which no priori knowledge concerning the zero point of  $Y$  is available. Since we assume that such a priori knowledge is not available, then any of the iterative methods will often fail to calculate the zero  $\bar{x}$ , because poor starting value is likely to be chosen. As a possible remedy, one defines a homotopy or deformation  $H: R \times R \rightarrow R$  such that

$$H(x, 1) = Q(x); \quad H(x, 0) = Y(x),$$

where  $Q: R \rightarrow R$  is a (trivial) smooth map having known zero point and  $H$  is also smooth. Typically, one may choose a convex

$$H(x, \lambda) = \lambda Q(x) + (1 - \lambda) Y(x), \quad (13)$$

and attempt to trace an implicitly defined curve  $\Phi(z) \in H^{-1}(0)$  from a starting point  $(x_1, 1)$  to a solution point  $(\bar{x}, 0)$ . If this succeeds, then a zero point  $\bar{x}$  of  $Y$  is obtained. The curve  $\Phi(z) \in H^{-1}(0)$  can be traced numerically if it is parameterized with respect to the parameter  $\lambda$ , then the classical embedding methods can be applied (Allgower and George 1990).

**3. Computational Developments**

**3.1 Recurrence formulae for the nth. derivative of  $L(z)$**

For the iterative formulae of Equation (1), we establish for the  $n$ th derivative of  $L(z)$  the following recurrence formula:

$$2zL^{(n+1)}(z) = 3C_1 G_1^{(n)}(z) + 3C_2 G_2^{(n)}(z) - (2n + 3)L^{(n)}, \quad (14)$$

where the  $G$ 's functions satisfy the recurrence formulae:

$$G_i^{(n)}(z) = 0.125(2n - 1)\lambda_i^2 (G_i^{(0)}(z))^2 G_i^{(n-1)}(z);$$

$$i = 1, 2; \quad m = 1, 2, \dots, k, \quad G_i^{(0)}(z) = \left(1 - \frac{z\lambda_i^2}{4}\right)^{-\frac{1}{2}}. \quad (15)$$

**3.2 Computational Algorithms**

Two algorithms are established in this section:

- The first one is for tracing the curve  $\Phi(z) \in H^{-1}(0)$  from  $\lambda = 1$  to  $\lambda = 0$ .
- The second one is for solving the basic equation of universal Lambert problem (1).

**3.2.1 Computational Algorithm 1**

- **Purpose:** To solve  $Y(x) = 0$  by embedding method.
- **Input:** (1) The function  $Q(x)$  with defined root  $x_1$  such that  $H(x_1, 1) = 0$   
(2) Positive integer  $m$ .
- **Output:** Solution  $x$  of  $Y(x) = 0$ .
- **Computational Sequence:**
  1. Set  $x = x_1, \lambda = (m - 1)/m, \Delta\lambda = 1/m$

2. For  $i = 1$  to  $m$  do  
 Begin  
 Solve  $H(y, \lambda) = \lambda Q(y) + (1 - \lambda) Y(y) = 0$ ,  
 iteratively for using  $x$  as starting value  
 $x = y; \quad \lambda = \lambda - \Delta\lambda$ .  
 End.

**3.2.2 Computational Algorithm 2**

- **Purpose:** To solve the basic equation of Lambert's theorem by iterative schemes of quadratic up to 1 th convergence orders without priori knowledge of the initial guess using homotopy continuation method with  $Q(z) = z + 1$ .
- **Input:**  $r_1, r_2, c, \Delta t, l, m$  (positive integer), Tol (specified tolerance),  $\mu$ .
- **Output:** The solution  $z$  of the basic equation of the universal Lambert's theorem.
- **Computational steps:**

1. Set  $z = -1; \Delta\lambda = \frac{1}{m}; \lambda = 1 - \Delta\lambda;$   
 $\lambda_1 = \sqrt{r_1 + r_2 + c}; \lambda_2 = \sqrt{r_1 + r_2 - c}$   
 $C_1 = \pm \frac{\lambda_1^3}{6K}; C_2 = \mp \frac{\lambda_2^3}{6K}; K = \sqrt{\mu};$   
 $C_3 = \begin{cases} 0 & \text{for case1} \\ \frac{-3\pi}{K} & \text{for case2} \end{cases}$

2. For  $i := 1$  to  $m$   
 Begin {i}  
 $q = 1 - \lambda; A_1 = F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z}{4} * \lambda_1^2\right);$   
 $A_2 = F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z}{4} * \lambda_2^2\right)$   
 $u = z^{-\frac{5}{2}}; FV = C_1 * A_1 + C_2 * A_2 - \frac{2}{3} * z * C_3 * u - \Delta t$   
 $L = \lambda * (z + 1) + q * FV;$   
 $S_1 = \left(1 - \frac{z}{4} * \lambda_1^2\right)^{\frac{1}{2}}; S_2 = \left(1 - \frac{z}{4} * \lambda_2^2\right)^{\frac{1}{2}}$   
 $Q_1 = \frac{3}{2z} (S_1 - A_1); Q_2 = \frac{3}{2z} (S_2 - A_2);$   
 $F[1] = C_1 Q_1 + C_2 Q_2 + C_3 Q_3.$   
 $Fa[1] = \lambda + q F[1]; \quad \Delta z = -\frac{L}{Fa[1]};$   
 If  $l = 2$  go to step 5  
 $G_1 = 0.125 \lambda_1^2 S_1^3; G_2 = 0.125 \lambda_2^2 S_2^3$   
 $F[2] = \frac{3}{2z} [3C_1 G_1 + 3C_2 G_2 - 5 F[1]];$   
 $Fa[2] = q F[2]$   
 $H = Fa[1] + \Delta z \frac{Fa[2]}{2}; \quad \Delta z = -\frac{L}{H}$   
 If  $l = 3$  go to step 3  
 For  $k = 1, 4$   
 Do:  $H = Fa[1]; B = 1; n = k - 1$   
 $G_1 = 0.125 (2n - 3) \lambda_1^2 S_1^2 G_1;$   
 $G_2 = 0.125 (2n - 3) \lambda_2^2 S_2^2 G_2.$

$$F[n] = \frac{1}{2z} [3C_1 G_1 + 3C_2 G_2 - (2n + 1)F[n - 1]];$$

$Fa[n] = q F[n]$   
 For  $j: 1$  to  $n - 1$  Do:  $B = \Delta z \frac{B}{j+1}; H = H + B Fa[j + 1]$   
 end {j} ;  $\Delta z = -\frac{L}{H};$  end {k}  
 3.  $z = z + \Delta z; \quad \lambda = \lambda - \Delta\lambda$   
 If  $|\Delta z| \leq \text{Tol}$  go to step 4;  
 go to step 2.  
 4. End  
 Finally, the accuracy of the computations could be checked by the following condition  $\epsilon = L - [\lambda (z + 1) + q FV].$

**3.3 Numerical Applications**

We used nine orbits with the initial and final position vectors and  $\Delta t$  listed in Table I, components of the position vectors are expressed in geocentric canonical unit ER ( ER=6378.1363 km) and time is expressed in unit of time TU (TU = 13.446849 solar minute).

Computational algorithm 2 is then applied for each orbit with  $\mu = 1, \text{Tol} = 10^{-8}$  and the computational check is satisfied within this tolerance, the solutions of the basic equation of the universal Lambert's problem and the check  $\epsilon$  are listed for each case in Table II.

Table I. Initial and final position vectors of the test orbits

Orbits	$x_1$	$y_1$	$z_1$	
$O_1$	0.702533	0.380583	-0.299824	
$O_2$	4.54359	-4.52863	-5.66845	
$O_3$	-0.483842	7.37763	1.49529	
$O_4$	0.303061	-0.168102	0.666923	
$O_5$	-0.0173445	-0.333483	-3.42012	
$O_6$	-4.70915	3.25104	3.52456	
$O_7$	1.72008	0.225063	0.674954	
$O_8$	-0.323474	-1.47143	-3.81656	
$O_9$	1.9652	0.72978	1.96275	
Orbits	$x_2$	$y_2$	$z_2$	$\Delta t$
$O_1$	0.702533	0.380583	-0.299824	1.22361
$O_2$	4.54359	-4.52863	-5.66845	2.90691
$O_3$	-0.483842	7.37763	1.49529	10.
$O_4$	0.303061	-0.168102	0.666923	1.5
$O_5$	-	-0.333483	-3.42012	2.28939
$O_6$	-4.70915	3.25104	3.52456	8.
$O_7$	1.72008	0.225063	0.674954	2.231
$O_8$	-0.323474	-1.47143	-3.81656	1.20888
$O_9$	1.9652	0.72978	1.96275	2.231

Table II. The solutions of the universal Lambert's problem for each test orbits

	s	Check
Orbit 1 elliptic	1.24687	$-2.22045 \times 10^{-16}$
Orbit 2 hyperbolic	-0.66667	0.
Orbit 3 parabolic	0.	$8.88178 \times 10^{-16}$
Orbit 4 elliptic	1.15666	$1.33227 \times 10^{-15}$
Orbit 5 hyperbolic	-1.24987	0.
Orbit 6 parabolic	0.	0.
Orbit 7 elliptic	1.02598	$4.44089 \times 10^{-16}$
Orbit 8 hyperbolic	-0.999998	$-1.11022 \times 10^{-16}$
Orbit 9 parabolic	0.	0.

### Conclusion

In concluding the present paper, an iterative method of arbitrary positive integer order of convergence  $\geq 2$  is developed for solving the basic equation of universal Lambert's problem. The method is characterized by: (1) it is of dynamic nature in the sense that, on going from one iterative scheme to the subsequent one, only additional instruction is needed, (2) it does not need any priori knowledge of the initial guess, a property which avoids it from falling in the critical situations between divergent to very slow convergent solution, that may exist in other numerical methods depending on initial guess as we mentioned before.

The algorithm based on powerful technique for solving transcendental equation without any priori knowledge of the initial guess, this technique is known as homotopy continuation method.

### References

- Allgower, E. L. and George, K. (1990). "Numerical Continuation Methods", Springer Verlag, Berlin
- Alshaary, A. A. (2003), General Algorithm for Homotopy Method of Arbitrary Order of Convergence for the Two-point Universal Initial Value of Space Dynamics, M. Sc Thesis Submitted to the Department of Mathematics, Girls College of Education, Jeddah-Saudi.
- Alshaary, A. A. (2008), New Analytical and Computational Developments for Orbital Boundary Value Problem –Lambert's Problem-of Space Dynamics, Ph. D Thesis Submitted to the Department of Mathematics, Girls College of Education, King Abdulaziz University, Jeddah-Saudi.
- Battin, R. H. (1964). "Astronautical Guidance", McGraw-Hill, New York.
- Danby, J. M. A. and Burkard, T. M. (1983), "Celestial Mechanics", 31, 95.
- Escobal, P. R. (1965). "Methods of Orbit Determination", Wiley, New York.
- Gooding, R. H. (1990). "A Procedure for the Solution of Lambert's Orbital Boundary Value Problem", Celestial Mechanics and Dynamical Astronomy, Vol. 48, pp 145-155.
- Herrick, S. (1971). "Astrodynamics", Vol. 1, Van Nostrand, London.
- Lancaster. E. R. and Blanchard, R. C. (1969). "A Unified Form of Lambert's Theorem", NASA Technical Note D-5368.
- Mansfield, R. L. (1989). "Uniform Extension of Gauss's Boundary Value Solution", Celestial Computing, Vol. 1, No. 2, Science and Engineering Software Co. Tarrytown, New York.
- Ralston, A. and Rabinowitz, P. (1978), "A First Course in Numerical Analysis", McGraw-Hill Kogakusha, Ltd. Tokyo.
- Sharaf, M. A. and Sharaf, A. A. (1998), Closest Approach In Universal Variables, Celestial Mechanics and Dynamical Astronomy, Vol. 69, pp 331.
- Sharaf, M. A., Banajh, A. A. and Alshaary, A. A. (2007), "Homotopy Continuation Method of Arbitrary Order of Convergence for Solving the Hyperbolic Form of Kepler Equation", J. Astrophysics and Astronomy, Vol 28, pp 9-16.
- Vallado, D. A. and McClain. W. C (2007) "Fundamentals of Astrodynamics and Applications ", 2nd Edition, Springer-Verlag.

11/27/2012