A Modified Gauss – Seidel Method for M - Matrices

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Abstract : in 1991, A. D. Gunawardena et al. proposed the modified Gauss-Seidel (MGS) method for solving the linear system with the preconditioned $= I + S$. The preconditioning Effect is not observed on the nth row. In the present paper, we suggest a new precondition. We get the convergence and comparison theorems for the proposed method. Numerical examples also given.

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1- Introduction:

We consider the following preconditioned linear system.

$$PAX = Pb, \tag{1.1}$$

where A = $(ai\ ,j)_{n\times n}{}^{\in R^{n\times n}}$ is a known non singular M-matrix, $P \in R^{n\times n}$, called the preconditioned, is non singular, $b \in R(A)$ is known and $X \in R^{n\times 1}$ is unknown, (A) is the range of A. throughout this paper, without loss of generality, we always assume that the coefficient matrix A has a splitting of the form $A = I - L - U$, where is the identity matrix, -L and -U are strictly lower triangular and strictly upper triangular parts of A, respectively.

To effectively solve the preconditioned linear system (1.1), a variety of preconditioners have been proposed by several authors, see [$1-6$] and the references there in. Since some preconditioned are constructed only from a part of upper triangular part of A, the preconditioning effect is not observed on the last row of matrix A. For example, the preconditioned $P_s = I + S$ presented in [1] and $P_{s_{max}} = I + S_{max}$ in [7] are formed respectively by

$$S = (s_{i,j}) = \begin{cases} -a_{i,i+1} & i = 1,2,\dots,n-1; \\ o, & Other\ wise \end{cases}$$

and

$$S_{max} = (s_{i,j}^m) = \begin{cases} -a_i, k_i, & i = 1,\dots,n-1, j > i; \\ o, & Other\ wise, \end{cases}$$

$$K_i = min\{j| \overset{max}{j} |a_{i,j}|, i < n\}$$

Motivated by their results, in this paper, we propose the following preconditioned:

$$P_m = I + S + R_{max} \tag{1.2}$$

where

$$(R_{max})_{i,j} = \begin{cases} -a_n, k_n, & i = n,\ j = K_n, \\ O, & Other\ Wise \end{cases}$$

With $K_n = min\{j||a_{n,j}| = max\{|a_{n,l}|, l = 1,\dots,n-1\}\}$

For the preconditioned (1.2), the preconditioned matrix

$$A_m = (I + S + R_{max})A$$

Can be split as

$$A_m = M_m - N_m$$

$= \left(I - D - L - E + R_{max} - \acute{D} - \acute{E}\right) - (U - S + SU),$

where D and E are respectively the diagonal, strictly lower triangular parts of SL, while $\acute{D}$ and $\acute{E}$ are the diagonal, strictly lower triangular, the MGS iterative matrix is $T_m = M_m^{-1} N_m$.

2. Preliminaries

For the convenience of the readers, we first give some of the notations, definitions and lemmas which will be used in what follows.

For $A = (a_{i,j}), B = (b_{i,j}) \in R^{n\times n}$, we write $A \geq B$ if $a_{i,j} \geq b_{i,j}$ holds for all $i,j = 1,2,\ldots,n$. $A \geq O$, called nonnegative, if $a_{i,j} \geq O$ for all $i,j = 1,2,\ldots,n$, where O is a $n \times n$ zero matrix. For the vectors $a, b \in R^{n\times 1}, a \geq b$ and $a \geq o$ can be defined in the similar manner.

Definition 2.1([9]). A matrix A is a L-matrix if $a_{i,i} > 0, i = 1,\ldots,n$ and $a_{i,j} \leq 0$ for all $i,j = 1,\ldots,n. i \neq j$. A nonsingular L-matrix A is a nonsingular M-matrix if $A^{-1} \geq 0$.

Lemma 2.1 ([10]). Let A be a nonnegative nonzero matrix. Then

(a)$\rho(A)$, the spectral radius of A, is an eigen value;

(b)A has a nonnegative eigenvector corresponding to $\rho(A)$;

(c) $\rho(A)$ is a simple eigen value of A;

(d) $\rho(A)$ increases when any entry of A increases.

Definition 2.2. Let A be a real matrix. Then, $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called

(a)regular if $M^{-1} \geq 0$ and $N \geq 0$[10];

(b)weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$[11];

(c)nonnegative if $M^{-1}N \geq 0$[12].

(d)M-splitting if M is a nonsingular M-,matrix and $N \geq 0$[13].

Definition 2.3 ([8]).We call $A = M - N$ the Gauss-Seidel splitting of A, if $M = (I - L)$ is nonsingular and $N = V$. In addition, the splitting is called

(a)Gauss-Seidel convergent if $\rho(M^{-1}N) < 1$;

(b)Gauss-Seidel regular if $M^{-1} = (I - L)^{-1} \geq 0$ and $N = U \geq 0$.

*Lemma*2.2([14]).Let $A = M - N$ Be an M-Splitting Of A. Then $\rho(M^{-1}N) < 1$ if and if A is a nonsingular M-matrix.

*Lemma*2.3([15]).Let A and B be $n \times n$ matrices. Then AB and BA have the same eigenvalues, counting multiplicity.

*Lemma*2.4([16]).Let A be a nonsingular M-matrix, and let $A = M_1 - N_1 = M_2 - N_2$ be two convergent splitting, the first one weak regular and the second one regular. If $M_1^{-1} \geq M_2^{-1}$, then $\rho(M_1^{-1}N_1 \leq \rho(M_2^{-1}N_2) < 1$.

3. Convergence And Comparison Theorems

We begin this section with a lemma given in [7].

For the preconditioned $P_S = I + S$, the Preconditioned Matrix $A_S = (I + S)A$ can be written as

$A_S = M_S - N_S = (I - D - L - E) - (U - S + SU)$.

In which D and E are defined as in section1. If $a_{i,i+1}a_{i+1,i} \neq 1 (i = 1,2,\ldots,n-1)$, then the MGS iterative matrix T_s for A_s can be defined by $T_s = M_S^{-1}N_S = (I - D - L - E)^{-1}(U - S + SU)$as $(I - D - L - E)^{-1}$ exists. In this case there is the following result:

Lemma 3.1 ([7]). Let $A = I - L - U$ be a nonsingular M-matrix. Assume that $O \leq a_{i,i+1}a_{i+1,i} < 1, 1 \leq i \leq n-1$, then $A_S = M_S - N_S$ is regular and Gauss-Seidel convergent.

Theorem 3.2. Let A be a nonsingular M-matrix. Assume that $O \leq a_{i,i+1}a_{i+1,i} < 1, 1 \leq i \leq n-1$ and $O \leq a_{n,k_j}a_{k_j,n} < 1, k_j = 1,\ldots,n-1$, then

$A_m = M_m - N_m$ regular and Gauss-Seidel convergent splitting.

$Proof$. We observe that when $O \le a_{i,i+1}a_{i+1,i} < 1, 1 \le i \le n-1$ and $O \le a_{n,k_j}a_{k_j,n} < 1, k_j = 1, \dots, n-1$, the diagonal elements of A_m are positive and M_m^{-1} exists. It is known that (see [11]) an L-matrix A is a non singular M-matrix if and only if there exists a positive vector y such that $> o$. By taking such, the fact that $I + S + R_{max} \ge 0$ implies $A_m y = (I + S + R_{max})A_y > 0$. Consequently, the L-matrix A_m is a nonsingular M-matrix, which means $A_m^{-1} \ge 0$.

We note that $L - R_{max} + E + \acute{E} \ge 0$ since $L \ge R_{max} \ge 0$.

When $0 \le a_{i,i+1}a_{i+1,i} < 1, 1 \le i \le n-1$ and $0 \le a_{n,k_j}a_{k_j,n} < 1, k_j = 1, \dots, n-1$, we have $D + \acute{D} < I$, so that $(I - D - \acute{D}) \ge 0$. Hence,

$$M_m^{-1} = [(I - D - \acute{D}) - (L - R_{max} + E + \acute{E})]$$

$$= [I - (I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E})]^{-1}(I - D - \acute{D})^{-1}$$

$$= [I + (I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E}) + [(I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E})]^2 + \cdots$$

$$+[(I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E})^{n-1}\}(I - D - \acute{D})^{-1}$$

$$\ge 0$$

On the other hand, it is to see that $N_m = U - S + SU \ge O$ since $U \ge S$ and $SU \ge 0$. Therefore, $A_m = M_m - N_m$ is a regular and Gauss-Seidel convergent splitting by definition 2.3 and lemma2.2.

For the splitting $A = I - L - U$ of matrix A, the iteration matrix of the classical Gauss-Seidel method for A is $T = (I - L)^{-1}U$. Comparing $\rho(T)$ with $\rho(T_m)$, the spectral radius of the MGS with the preconditioned$P_m = I + S + R_{max}$, we have the following comparison theorem:

$Theorem$ 3.3. Let A be a nonsingular M-matrix. Then under the assumptions of theorem3.2, we have $\rho(T_m) \le \rho(T) < 1$.

$Proof$. For $M_m = I - D - L - E + R_{max} - \acute{D} - \acute{E}$ and $N_m = U - S + SU$, by theorem 3.2, we know that $A_m = P_m A = M_m - N_m$ is a Gauss-Seidel convergent splitting. Since A is a nonsingular, the classic Gauss-Seidel splitting $A = (I - L) - U$ of A is clearly regular and convergent.

To compare $\rho(T_m)$ with $\rho(T)$, we consider the following splitting of A:

$$A = (I + S + R_{max})^{-1}M_m - (I + S + R_{max})^{-1}N_m$$

If we take $M_1 = (I + S + R_{max})^{-1}M_m$and $N_1 = (I + S + R_{max})^{-1}N_m$, then $\rho(M_1^{-1}N_1) < 1$ since $M_1^{-1}N_m = M_1^{-1}N_1$.

Also note that

$$M_1^{-1} = (I - D - L - E + R_{max} - \acute{D} - \acute{E})^{-1}(I + S + R_{max})$$

$$\ge (I - D - L - E + R_{max} - \acute{D} - \acute{E})^{-1}$$

$$= [I - (I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E})]^{-1}(I - D - \acute{D})^{-1}$$

$$\ge [I - (I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E})]^{-1}$$

$$\ge (I - L)^{-1},$$

It follows from lemma 2.4 that $\rho(M_1^{-1}N_1) \le \rho(M^{-1}N) < 1$.

Hence$\rho(M_m^{-1}N_m) \le \rho(M^{-1}N) < 1, i.e., \rho(T_m) \le \rho(T) < 1$.

Next, we give a comparison theorem between the MGS methods with the preconditioners P_m and P_s respectively.

$Theorem$ 3.4. Let A be a nonsingular M-matrix. Then under the assumptions of theorem 3.2 and $a_{k_n}j \le a_{k_n}, n\, a_n, j, 1 \le n-1$, we have$\rho(T_m) \le \rho(T_s) < 1$.

$Proof$. For the matrices M_s,M_m,N_s and N_m in the splitting of matrices $P_s A = M_s - N_s$ and $P_m A = M_m - N_m$, they can be expressed in the partitioned forms as follows:

$$M_s = I - D - L - E = \left(\begin{array}{c|c} \hat{M} & 0 \\ U^T & 1 \end{array}\right),$$

$$M_m = I - D - L - E + R_{max} - \acute{D} - \acute{E},$$

$$M_m = M_s + R_{max} \times A = \left(\begin{array}{c|c} \widehat{M} & 0 \\ V^T & U_n \end{array}\right),$$

$$N_m = N_s = \left(\begin{array}{c|c} \widehat{N} & W \\ 0 & 0 \end{array}\right),$$

where

$$\widehat{M} = (\widehat{m}_{i,j}), \widehat{m}_{i,j} = \begin{cases} 0, 1 \le i \le j \le n-1 \\ 1 - a_{i,i+1}a_{i+1,i}, i = j, \\ a_{i,j} - a_{i,i+1}a_{i+1,j}, j < i \le n-1, \end{cases}$$

$$U^T = (a_{n,1}, \dots, a_{n,n-1}),$$

$$V^T = (V_1, \dots, V_{n-1}), V_j = a_{n,j} - a_{n,kn}a_{kn,j} (1 \le j \le n-1),$$

$$V_n = 1 - a_{n,kn}a_{kn,n},$$

$$W = (W_1, \dots, W_{n-1})^T, W_i = -a_{i,n} + a_{i,i+1}a_{i+1,n} (1 \le I \le n-1),$$

and $\widehat{N} \ge 0$ is an $(n-1) \times (n-1)$ strictly upper triangular matrix.

Direct computation yields

$$M_s^{-1} = \left(\begin{array}{c|c} \widehat{M}^{-1} & 0 \\ \hline U^T\widehat{M}^{-1} & 1 \end{array}\right) \quad \text{and}$$

$$M_m^{-1} = \left(\begin{array}{c|c} \widehat{M}^{-1} & 0 \\ \hline -V_n^{-1}V^T\widehat{M}^{-1} & V_n^{-1} \end{array}\right).$$

Therefore,

$$N_s M_s^{-1} = \left(\begin{array}{c|c} \widehat{T}_s & W \\ 0 & 0 \end{array}\right) \ge 0$$

and

$$N_m M_m^{-1} = \left(\begin{array}{c|c} \bar{T}_m & U_n^{-1}W \\ 0 & 0 \end{array}\right) \ge 0$$

where $\widehat{T}_s = \widehat{N}\widehat{M}^{-1} - Wu^T\widehat{M}^{-1}$ and $\bar{T}_m = \widehat{N}\widehat{M}^{-1} - WV_n^{-1}V^T\widehat{M}^{-1}$.

Obviously, $\rho(N_s M_s^{-1}) = \rho(\widehat{T}_s)$ and $\rho(N_m M_m^{-1}) = \rho(\bar{T}_m)$.

By simple computation, we know $\bar{T}_m \le \widehat{T}_s$ that under the assumption $a_{kn,j} \le a_{kn,n}a_{n,j}, 1 \le j \le n-1$. Hence by lemma 2.1, we have

$$\rho(N_m M_m^{-1}) = \rho(\bar{T}_m) \le \rho(\widehat{T}_s) = \rho(N_s M_s^{-1})$$

Therefore, by lemma 2.3, we immediately know that which means that $\rho(T_m) \le \rho(T_s)$.

4. Numerical Examples

In this part, we give some examples to illustrate the theory in section3.

Example 4.1.Let us consider the matrix A of (1.1), given by

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}.$$

We have $\rho(T_m) = 0.3114$ and $\rho(T_s) = 0.3384$. Clearly, $\rho(T_m) < \rho(T_s)$ holds.

Example 4.2. Let the coefficient matrix A of (1.1) be given by

$$A = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.3 & 1 & -0.6 \\ -0.3 & -0.3 & 1 \end{pmatrix}.$$

We have $\rho(T_m) = 0.29167 < \rho(T_s) = 0.44763$

References:

[1] A. Berman, R. J. Plemmons, Nonnegative matrices in the mathematical sciences, Academic Press, New York, 1979.

[2] A. D. Gunawardena, S. K. Jain, L. S. Nyder, Modified iterative methods for consistent linear system, Linear Algebra Appl. $154-156(1991)123-143$.

[3] T. Kohno, M. Kotakemori, H. Niki, Improving the modified Gauss-Seidel method for Z-matirices, Linear Algebra Appl. 267 $(1997)113-123$.

[4] H. Kotakemori, K. Haradu, M. Morimoto, H. Niki, A comparison theorem for the iterative method with the preconditioned $(I + S_{max})$, J. Comput. Appl. Math. 145 $(2002)373-378$.

[5] H. Kotakemori, H. Niki, N. Okamoto, Accelerated iterative method for Z-matrices, J. Comput. Appl. Math.75(1996)87 − 97.

[6] W. Li, W. W. Sun, Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices, Linear Algebra Appl. 317(2000)227 − 240.

[7] W. Li, A note on the preconditioned Gauss-Seidel (GS) method for linear system, J. Comput. Appl. Math. 182 (2005)81 − 90.

[8] J. P. Milaszewicz, Impriving Jacobi and Gauss-Seidel iterations, Linear Algebra Appl.93(1987)161 − 170.

[9]H. Niki, K. Harada, M. Morimoto, M. Sakakihara, The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method, J. Comput. Appl. Math. 164 − 165 (2004)587 − 600.

[10] H. Niki, T. Kohno, M. Morimoto, The preconditioned Gauss-Seidel method faster than the SOR method, J. Comput. Appl. Math. 219(2008)59 − 71.

[11] H. Schneider, Theorems on M-splitting of a singular M-matrix which depend on graph structure, Linear Algebra Appl. 58(1984)407 − 424.

[12] Y. Z. Song, Comparisons of non negative splitting of matrices, Linear Algebra Appl.154 − 156(1991)433 − 455.

[13] R. S. Varga, Matrix iterative analysis, Prentice-Hall, Englewood Cliffs, NJ, 1981.

[14] Z. I. Wozniki, Nonnegative splitting theory, Japan J. Industrial Appl. Math. 11(1994)289 − 342.

[15] D. M. Young, Iterative solution of large linear systems, Academic Press, New York,1971.

[16] F. Zhang, Matrix theory, Springer, 1999.

[17] Bing Zheng, S. X. Miao, Two new modified Gauss-Seidel methods for linear system with m-Matrices, J. Comput. Appl. Math. Vol. 233 Issue 4, (2009) 922-930.