Chebyshev Galerkin method for approximate solution of a class of Fredholm hypersingular integral equations

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Abstract: A simple galerkin method based on second type Chebyshev polynomials approximation method is employed to obtain approximate solution of a class of hypersingular integral equations of the second kind. For a class of hypersingular integral equation of the second kind, this method avoids the complex function-theoretic, long computations of collocation polynomial-based methods and produces the known exact solution or high accurate approximate solution.

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1 Introduction

Hypersingular integral equation is considered as an important tool in applied Mathematics as it finds application in solving a large class of mixed boundary value problems arising in mathematical physics. Particularly the crack problems in fracture mechanics or water wave scattering problems involving barriers, diffraction of electromagnetic waves and aerodynamics problems ([2,6,7]) could be reduced to hypersingular integral equations in single or disjoint multiple intervals.

A simple approximation method for solving a general hypersingular integral equation of the first kind where the kernel consists a hypersingular part and a regular part is introduced and developed in [10]. A method based on polynomial approximation is used in [9] to produce the approximate solution of a class of singular integral equations of the second kind. Dutta and Banerjea ([4]) have solved a hypersingular integral equation in two intervals by using the solution of Cauchy type singular integral equations in two disjoint intervals. Gori et.al. ([5]) have constructed a quadrature rule based on the use of suitable refinable quasi-interpolatory operators, for the numerical evaluation of Hadamard finite-part integrals. Chen and Zhou ([3]) have developed an efficient method for solving hypersingular integral equation of the fist kind in reproducing kernel space in order to eliminate the singularity of the equation.

In the present paper, we have considered the following two cases for hypersingular integral equation of the second kind.

Case A: We consider the following Fredholm hypersingular integral equation of the second kind ([9])

$$u(x)=f(x)+\frac{\alpha(1-x^2)^{1/2}}{\pi}\int_{-1}^{1}\frac{u(t)}{(t-x)^2}dt, \qquad -1<x<1 \quad (1\text{-}1)$$

on the finite interval $(-1,1)$ with the condition $u(\pm 1)=0$. Equation (1-1) is generalized state for oval wing of Prandtl's equation, where $\alpha>0$ is a known value, $f(x)$ and $u(x)$ are known and unknown functions respectively. Equation (1-1) is referred to as Hadamard finite part ([9])

$$\int_{-1}^{1}\frac{u(t)}{(t-x)^2}dt = \lim_{\varepsilon\to 0^+}\left[\int_{-1}^{x-\varepsilon}\frac{u(t)}{(x-t)^2}dt+\int_{x+\varepsilon}^{1}\frac{u(t)}{(x-t)^2}dt-\frac{u(x+\varepsilon)+u(x-\varepsilon)}{\varepsilon}\right]. \quad (1\text{-}2)$$

Accurate solution of equation (1-1) was obtained by using a simple approximating polynomial for $u(x)$ in [9] and by reducing it to a differential problem of Riemann-Hilbert on the interval $(-1,1)$ in [1].

Case B: We also consider the following Fredholm hypersingular integral equation of the first kind

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{(x-t)^2}dt = f(x), \qquad -1<x<1. \qquad (1\text{-}3)$$

Equation (1-3) has been solved in a closed form in [8] as follows

$$u(x)=\frac{1}{\pi}\int_{-1}^{1}f(t)\ln\left|\frac{x-t}{1-xt+\{(1-x^2)(1-t^2)\}^{1/2}}\right|dt, \qquad -1<x<1. \quad (1\text{-}4)$$

The integral occurred in (1-4) is not straightforward for evaluation.

In this paper the equations (1-1) and (1-3) are solved using a Chebyshev Galerkin method. The numerical approximations satisfy with the exact solutions offered in [1,8,9].

2 Methods

Case A: In equation (1-1) it is assumed that

$$u(x)=(1-x^2)^{1/2}\psi(x), \qquad (2\text{-}1)$$

where $\psi(x)$ is a smooth function (see [8]). Let us approximate $\psi(x)$ with a truncated series as follows

$$\psi(x)\approx\sum_{j=0}^{n}a_jU_j(x), \qquad (2\text{-}2)$$

where $U_j(x)$, $j=0,1,\dots,n$ are Chebyshev polynomials of the second type defined as

$$U_j(x)=\frac{\sin(j+1)\theta}{\sin\theta}, \qquad j=0,1,\dots,n, \qquad x=\cos\theta. \quad (2\text{-}3)$$

The Chebyshev polynomials of the second type satisfy the following recursive relation

$$\begin{aligned} &U_0(x)=1,\\ &U_1(x)=2x, \qquad (2\text{-}4)\\ &U_{n+1}(x)=2xU_n(x)-U_{n-1}(x), \qquad n=1,2,\dots \end{aligned}$$

These polynomials satisfy the following orthogonality property

$$\int_{-1}^{1}(1-x^2)^{1/2}U_i(x)U_j(x)dx=\begin{cases}\dfrac{\pi}{2}, & i=j\\[2ex] 0, & i\neq j\end{cases} \qquad (2\text{-}5)$$

After substituting (2-1) in (1-1) and getting simplification we have

$$\sum_{j=0}^{n}a_j\left[U_j(x)-\frac{\alpha}{\pi}\int_{-1}^{1}\frac{(1-t^2)^{1/2}U_j(t)}{(t-x)^2}dt\right]=\frac{f(x)}{(1-x^2)^{1/2}}. \quad (2\text{-}6)$$

Refereing to [9] we have

$$\begin{aligned}\frac{1}{\pi}\int_{-1}^{1}\frac{(1-t^2)^{1/2}U_j(t)}{(t-x)^2}dt &= \frac{1}{\pi}\frac{d}{dx}\int_{-1}^{1}\frac{(1-t^2)^{1/2}U_j(t)}{(t-x)}dt \qquad (2\text{-}7)\\ &= -(j+1)U_j(x).\end{aligned}$$

With substituting (2-7) in (2-6) we get

$$\sum_{j=0}^{n}a_j\left[1+\alpha(j+1)\right]U_j(x)=\frac{f(x)}{(1-x^2)^{1/2}}. \qquad (2\text{-}8)$$

By multiplying (2-8) in $(1-x^2)^{1/2}U_i(x)$, $(i=0,1,\dots,n)$, integrating on the interval $(-1,1)$ and using the orthogonality property (2-5) we get

$$a_i=\frac{2f_i}{\pi(1+\alpha(i+1))}, \qquad i=0,1,\dots,n \qquad (2\text{-}9)$$

where

$$f_i = \int_{-1}^{1} f(x) U_i(x) dx, \quad i = 0,1,\ldots,n. \qquad (2\text{-}10)$$

Case B: We can use the same approach to solve equation (1-3). With substituting (2-1) and (2-2) in (1-3) we have

$$\sum_{j=0}^{n} a_j \frac{1}{\pi} \frac{d}{dx} \int_{-1}^{1} \frac{(1-t^2)^{1/2} U_j(t)}{t-x} dt = f(x), \quad (2\text{-}11)$$

and using (2-7) yields

$$-\sum_{j=0}^{n} a_j (j+1) U_j(x) = f(x), \qquad (2\text{-}12)$$

By multiplying (2-7) in $(1-x^2)^{1/2} U_i(x)$, $(i = 0,1,\ldots,n)$, integrating on the interval $(-1,1)$ and using the orthogonality property (2-5) we get

$$a_i = -\frac{2F_i}{(i+1)\pi}, \qquad i = 0,1,\ldots,n \qquad (2\text{-}13)$$

where

$$F_i = \int_{-1}^{1} (1-x^2)^{1/2} f(x) U_i(x) dx, \quad i = 0,1,\ldots,n. \quad (2\text{-}14)$$

3 Numerical Illustrations

Example 1: In the hypersingular integral equation (1-1) assuming

$$f(x) = 2\pi(1-x^2)^{1/2}, \qquad \alpha = \pi. \qquad (3\text{-}1)$$

With (2-9) and (2-10) we get

$$a_0 = \frac{4\pi}{\pi+2}, \; a_j = 0, \; j = 1,2,\ldots \qquad (3\text{-}2)$$

Thus from (2-2) we get

$$\psi(x) = \frac{4\pi}{\pi+2}, \quad (3\text{-}3)$$

and finally we get the approximate solution of the integral equation from (2-1) as

$$u_{app}(x) = \frac{4\pi}{\pi+2} (1-x^2)^{1/2}, \qquad (3\text{-}4)$$

which is the exact solution of the equation.

Example 2: In the hypersingular integral equation (1-3) assume $f(x) = x^2$. In this case we get the exact solution with (1-4) as follows

$$u(x) = -\left(\frac{1}{3}x^2 + \frac{1}{6}\right)(1-x^2)^{1/2}, \qquad (3\text{-}5)$$

With (2-9) and (2-10) we get

$$a_0 = -\frac{1}{4}, \; a_2 = -\frac{1}{12}, \; a_j = 0, \; j \neq 0,2 \qquad (3\text{-}6)$$

then from (2-2) we get

$$\psi(x) = -\frac{1}{4} U_0(x) - \frac{1}{12} U_2(x) = -\left(\frac{1}{3}x^2 + \frac{1}{6}\right),$$

substituting in (2-1) which gives the same exact solution (3-5).

Example 3: In (1-3) assume $f(x) = \sin x$. For $n = 5$ we get

$$a_0 = a_2 = a_4 = 0,$$

$$a_1 = -0.2298069699, \; a_3 = 0.004953277928, \; a_5 = -0.000041876676,$$

then

$$\begin{aligned} u_{app}(x) &= (1-x^2)^{1/2}\sum_{j=0}^{5} a_j U_j(x) \\ &= (1-x^2)^{1/2}(-0.4796783115x + 0.0409662771x^3 - .0013400536x^5) \end{aligned} \tag{3-7}$$

In this case the exact solution is not straightforward because the integral (1-4) is difficult. We evaluate $u(x)$ in (1-4) for any $x \in [-1,1]$, numerically. Table 1 shows absolute errors of the solution obtained by Chebyshev galerkin method comparing with the results obtained from (1-4) for different values of n at some $x \in [-1,1]$.

Absolute errors for Example 3

x	n=5	n=10	n=15	n=20
1	0	0	0	0
-0.8	0.17E-06	0.99E-12	0.23E-13	0.49E-27
-0.6	0.17E-06	0.99E-12	0.23E-13	0.35E-27
-0.4	0.29E-07	0.97E-12	0.90E-14	0.71E-28
-0.2	0.19E-06	0.66E-12	0.24E-14	0.22E-27
0	0	0	0	0
0.2	0.19E-06	0.66E-12	0.24E-14	0.22E-27
0.4	0.29E-07	0.97E-12	0.90E-14	0.71E-28
0.6	0.17E-06	0.99E-12	0.23E-13	0.35E-27
0.8	0.17E-06	0.99E-12	0.23E-13	0.49E-27
1	0	0	0	0

4 Conclusion

We have computed the approximate solution of hypersingular integral equations of the second kind easily and carefully. Our method have avoided the complex function-theoretic long computations.

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