Non-Polynomial Spline Approach to the Solution of Twelfth-Order Boundary-Value Problems

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Abstract: Non-polynomial spline in off step points is used to solve special twelfth order linear boundary value problems. Associated boundary formulas are developed. Truncation errors are given. Two examples are considered for the numerical illustration. However, it is observed that our approach produce better numerical solutions in the sense that max $|e_i|$is minimum.

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1 Introduction

We consider twelfth -order boundary-value problem of type

$$y^{(12)}(x)+f(x)y(x)=g(x) \quad , \quad x\in[a,b] \qquad (1)$$

With boundary conditions

$$y(a)=\alpha_0, y^{(1)}(a)=\alpha_1, y^{(2)}(a)=\alpha_2, y^{(3)}(a)=\alpha_3, y^{(4)}(a)=\alpha_4, y^{(5)}(a)=\alpha_5,$$
$$y(b)=\beta_0, y^{(1)}(b)=\beta_1, y^{(2)}(b)=\beta_2, y^{(3)}(b)=\beta_3, y^{(4)}(b)=\beta_4, y^{(5)}(b)=\beta_5 \qquad (2)$$

where α_i , β_i for $i=0,1,2,3,4,5$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$.

The solutions of twelfth-order boundary value problems are not very much found in the numerical analysis literature. These problems generally arise in the mathematical modelling of viscoelastic flows [1, 2].

The conditions for existence and uniqueness of solution of such boundary value problems are explained by theorems presented in Agarwal [3]. Siddiqi and Twizell [4-7] presented the solutions of sixth-, eighth-, tenth- and twelfth-order boundary value problems using the sixth, eighth, tenth and twelfth degree spline, respectively. Siddiqi and Ghazla [8] Solutions of 12th order boundary value problems using non-polynomial spline technique.

Following the spline functions proposed in this paper have the form:

$$T_{13}=\mathrm{span}\{1,x,x^2,x^3,x^4,x^5,x^6,x^7,x^8,x^9,x^{10},x^{11},\cos(kx),\sin(kx)\}$$

where k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \le x \le x_{i+1}$ we have

$$\mathrm{span}\{1,x,x^2,x^3,x^4,x^5,x^6,x^7,x^8,x^9,x^{10},x^{11},\cos(|k|x),\sin(|k|x)\}$$

In this paper, in Section 2, the new Non-polynomial spline methods are developed for solving equation (1) along with boundary condition (2). Development of the boundary formulas are considered in Section 3 and in Section 4 numerical experiment, discussion are given:

2 Numerical methods

To develop the spline approximation to the twelfth -order boundary-value problem (1)-(2), the interval [a,b] is divided in to n equal subintervals using the grid $x_{i-\frac{1}{2}}=a+(i-\frac{1}{2})h$, $i=1,2,3,..,n$ where $h=\dfrac{b-a}{n}$. We Consider the following Non-polynomial spline $S_i(x)$ on each Subinterval $[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]$, $i=0,1,2,..,n-1$, $x_0=a$, $x_n=b$,

$$S_i(x)=a_i\cos k(x-x_i)+b_i\sin k(x-x_i)+c_i(x-x_i)^{11}+d_i(x-x_i)^{10}+e_i(x-x_i)^9+$$
$$f_i(x-x_i)^8+g^*_i(x-x_i)^7+o_i(x-x_i)^6+p_i(x-x_i)^5+q^*_i(x-x_i)^4+u_i(x-x_i)^3+$$
$$z_i(x-x_i)^2+r_i(x-x_i)+t_i^*$$
(3)

where $a_i, b_i, c_i, d_i, e_i, f_i, g^*{}_i, o_i, p_i, q_i^*, u_i, z_i, r_i,$ and t_i^* are real finite constants and k is free parameter.

The spline is defined in terms of its 1td, 2th, 3th, 4th, 5th and 12th derivatives and we denote these values at knots as:

$$S_i(x_{i-\frac{1}{2}}) = y_{i-\frac{1}{2}}, S_i'(x_{i-\frac{1}{2}}) = m_{i-\frac{1}{2}}, S_i''(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}}, S_i'''(x_{i-\frac{1}{2}}) = z_{i-\frac{1}{2}}, S_i^{(4)}(x_{i-\frac{1}{2}}) = V_{i-\frac{1}{2}},$$
$$S_i^{(5)}(x_{i-\frac{1}{2}}) = w_{i-\frac{1}{2}}, S_i^{(12)}(x_{i-\frac{1}{2}}) = L_{i-\frac{1}{2}},$$
$$S_i(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}}, S_i'(x_{i+\frac{1}{2}}) = m_{i+\frac{1}{2}}, S_i''(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}}, S_i'''(x_{i+\frac{1}{2}}) = z_{i+\frac{1}{2}}, S_i^{(4)}(x_{i+\frac{1}{2}}) = V_{i+\frac{1}{2}}$$
$$S_i^{(5)}(x_{i-\frac{1}{2}}) = w_{i-\frac{1}{2}}, \ S_i^{(12)}(x_{i+\frac{1}{2}}) = L_{i+\frac{1}{2}}$$
$$\text{For } i = 0,1,2,\ldots,n-1. \tag{4}$$

Assuming $y(x)$ to be the exact solution of the boundary value problem (1) and y_i be an approximation to $y(x_i)$ using the continuity conditions ($S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ where $\mu = 6,7,8,9,10$ and 11), after lengthy calculations we obtain the following spline relations:

$$y_{i-\frac{13}{2}} - 12y_{i-\frac{11}{2}} + 66y_{i-\frac{9}{2}} - 220y_{i-\frac{7}{2}} + 495y_{i-\frac{5}{2}} - 792y_{i-\frac{3}{2}} + 924y_{i-\frac{1}{2}} - 792y_{i+\frac{1}{2}} + 495y_{i+\frac{3}{2}} +$$
$$220y_{i+\frac{5}{2}} + 66y_{i+\frac{7}{2}} - 12y_{i+\frac{9}{2}} + y_{i+\frac{11}{2}} = h^{10}[\alpha L_{i-\frac{13}{2}} + \beta L_{i-\frac{11}{2}} + \gamma L_{i-\frac{9}{2}} + \delta L_{i-\frac{7}{2}} + \eta L_{i-\frac{5}{2}} + \rho L_{i-\frac{3}{2}} +$$
$$\tau L_{i-\frac{1}{2}} + \rho L_{i+\frac{1}{2}} + \eta L_{i+\frac{3}{2}} + \delta L_{i+\frac{5}{2}} + \gamma L_{i+\frac{7}{2}} + \beta L_{i+\frac{9}{2}} + \alpha L_{i+\frac{11}{2}}] \quad , \quad i = 7,8,\ldots,n-7.$$
$$(5)$$

Where

$$\alpha = \frac{\csc\theta}{39916800\theta^{12}}(-39916800\,\theta + 6652800\,\theta^3 - 332640\,\theta^5 + 7920\,\theta^7 - 110\theta^9 + \theta^{11} +$$
$$39916800\,\mathrm{Sin}\,\theta),$$
$$\beta = \frac{-\csc\theta}{19958400\theta^{12}}(\theta\,(-39916800 + 6652800\,\theta^2 - 332640\,\theta^4 + 7920\,\theta^6 - 110\,\theta^8 +$$
$$\theta^{10})\,\mathrm{Cos}(\theta) + 2(-99792000\,\theta + 6652800\theta^3 + 1663200\,\theta^5 - 229680\,\theta^7 + 13750\,\theta^9 -$$
$$509\,\theta^{11} + 119750400\,\mathrm{Sin}(\theta))),$$
$$\gamma = \frac{-\csc\theta}{19958400\theta^{12}}(918086400\,\theta + 6652800\,\theta^3 - 12307680\,\theta^5 - 2843280\,\theta^7 + 748330\,\theta^9 -$$
$$76319\,\theta^{11} + 4\,\theta(99792000 - 6652800\,\theta^2 - 1663200\,\theta^4 + 229680\,\theta^6 - 13750\,\theta^8 +$$
$$509\,\theta^{10})\,\mathrm{Cos}(\theta) - 1317254400\,\mathrm{Sin}(\theta)),$$
$$\delta = -\frac{\csc\theta}{19958400\theta^{12}}(3\theta\,(-598752000 - 6652800\,\theta^2 + 8316000\,\theta^4 + 1892880\,\theta^6 -$$
$$498850\,\theta^8 + 50879\,\theta^{10})\,\mathrm{Cos}(\theta) + 2\,(-1297296000\,\theta - 73180800\,\theta^3 +$$
$$1663200\,\theta^5 + 3001680\,\theta^7 + 1650550\,\theta^9 - 551381\,\theta^{11}] + 2195424000\,\mathrm{Sin}(\theta)),$$

$$\eta = \frac{-\csc\theta}{133305600\theta^{12}}(3392928000\,\theta + 286070400\,\theta^3 - 28274400\,\theta^5 + 2874960\,\theta^7 +$$
$$290950\theta^9 - 3296917\,\theta^{11} + 64\,\theta\,(49896000 + 3326400\,\theta^2 - 134640\,\theta^6 -$$
$$68200\,\theta^8 + 22953\,\theta^{10}\})\,\mathrm{Cos}(\theta) - 6586272000\,\mathrm{Sin}(\theta)),$$

$$\rho = \frac{-1}{33264000\theta^{12}}(2634508800 + (\theta\,(-1397088000(1+\theta^2) - 18295200\,\theta^4 -$$
$$2383920\,\theta^6 + 103950\,\theta^8 + 1623019\,\theta^{10})\,\mathrm{Cot}(\theta)) + -2\,\theta\,(-618710400 +$$
$$59875200\,\theta^2 + 7650720\,\theta^4 + 1180080\,\theta^6 + 350130\,\theta^8 + 746989\,\theta^{10})\,\mathrm{Csc}(\theta)),$$

$$\tau = \frac{-1}{33264000\theta^{12}}(-3073593600 + 4\,\theta\,(419126400 + 46569600\,\theta^2 + 7650720\,\theta^4 +$$
$$1718640\,\theta^6 + 622930\,\theta^8 + 655177\,\theta^{10})\,\mathrm{Cot}(\theta) + \backslash\,\mathrm{theta}\,(1397088000 +$$
$$139708800\,\theta^2 + 18295200\,\theta^4 + 2383920\,\theta^6 - 103950\,\theta^8 - 1623019\,\theta^{10})\,\mathrm{Csc}(\theta)).$$

If $\theta \to 0$ then

$$(\alpha,\beta,\gamma,\delta,\eta) \to (\frac{1}{6227020800},\frac{1363}{1037836800},\frac{82207}{345945600},\frac{9106698}{124540416},\frac{28218769}{415134720}),$$
$$(\rho,\tau) \to (\frac{125468459}{518918400},\frac{27085381}{74131200}).$$

And the consistency relation of non-polynomial is reduced to consistency relation of the thirteenth polynomial spline functions derived in [9]. The local truncation error corresponding to the method equation (5) can be obtained as:

$$\begin{aligned}
t_i = {} & (1 - 2(\alpha+\beta+\gamma+\delta+\eta+\rho) - \tau)h^{12} y_i^{(12)} + (\frac{-1}{2} + \alpha+\beta+\gamma+\delta+\eta+\rho+\frac{\tau}{2})h^{13} y_i^{(13)} \\
& + (\frac{5}{8} - \frac{1}{4}(145\alpha + 101\beta + 65\gamma + 37\delta + 17\eta + 5\rho + \frac{\tau}{2}))h^{14} y_i^{(14)} \\
& + (\frac{-13}{48} + \frac{1}{24}(433\alpha + 301\beta + 193\gamma + 109\delta + 49\eta + 13\rho + \frac{\tau}{2}))h^{15} y_i^{(15)} \\
& + (\frac{119}{640} - \frac{1}{192}(21601\alpha + 10601\beta + 4481\gamma + 1513\delta + 353\eta + 41\rho + \frac{\tau}{2}))h^{16} y_i^{(16)} \\
& + (\frac{91}{1280} + \frac{1}{1920}(105121\alpha + 51001\beta + 21121\gamma + 6841\delta + 1441\eta + 121\rho + \frac{\tau}{2}))h^{17} y_i^{(17)} \\
& + (\frac{6821}{193536} - \frac{1}{23040}(3299185\alpha + 1151501\beta + 324545\gamma + 6637\delta + 8177\eta + 365\rho + \frac{\tau}{2})))h^{18} y_i^{(18)} \\
& + \ldots
\end{aligned}$$
$$i = 7,8,\ldots,n-7 \qquad (6)$$

3 Development of the boundary formulas

Liner system equation (5) consist of (n − 1) unknown, so that to obtain unique solution we need twelfth more equations to be associate with equation (5) so that we can develop the boundary formulas of different orders, but for sake of briefness here we develop the twelfth order boundary formulas so that we define the following identity:

$$w_0^{'}y_0+\sum_{i=0}^{8}a_i^{'}y_{i+\frac{1}{2}}+c^{'}hy_0^{'}+d^{'}h^2y_0^{''}+e^{'}h^3y_0^{'''}+u^{'}h^4y_0^{(4)}+p^{'}h^5y_0^{(5)}=h^{12}\sum_{i=0}^{10}b_i^{'}y_{i+\frac{1}{2}}^{(12)} \quad (7)$$

$$w_0^{''}y_0+\sum_{i=0}^{9}a_i^{''}y_{i+\frac{1}{2}}+c^{''}hy_0^{'}+d^{''}h^2y_0^{''}+e^{''}h^3y_0^{'''}+u^{''}h^4y_0^{(4)}+p^{''}h^5y_0^{(5)}=h^{12}\sum_{i=0}^{11}b_i^{''}y_{i+\frac{1}{2}}^{(12)} \quad (8)$$

$$w_0^{'''}y_0+\sum_{i=0}^{10}a_i^{'''}y_{i+\frac{1}{2}}+c^{'''}hy_0^{'}+d^{'''}h^2y_0^{''}+e^{'''}h^3y_0^{'''}+u^{'''}h^4y_0^{(4)}+p^{'''}h^5y_0^{(5)}=h^{12}\sum_{i=0}^{12}b_i^{'''}y_{i+\frac{1}{2}}^{(12)} \quad (9)$$

$$w_0^{\circ}y_0+\sum_{i=0}^{11}a_i^{\circ}y_{i+\frac{1}{2}}+c^{\circ}hy_0^{'}+d^{\circ}h^2y_0^{''}+e^{\circ}h^3y_0^{'''}+u^{\circ}h^4y_0^{(4)}+p^{\circ}h^5y_0^{(5)}=h^{12}\sum_{i=0}^{13}b_i^{\circ}y_{i+\frac{1}{2}}^{(12)} \quad (10)$$

$$w_0^{\circ\circ}y_0+\sum_{i=0}^{12}a_i^{\circ\circ}y_{i+\frac{1}{2}}+c^{\circ\circ}hy_0^{'}+d^{\circ\circ}h^2y_0^{''}+e^{\circ\circ}h^3y_0^{'''}+u^{\circ\circ}h^4y_0^{(4)}+p^{\circ\circ}h^5y_0^{(5)}=h^{12}\sum_{i=0}^{14}b_i^{\circ\circ}y_{i+\frac{1}{2}}^{(12)} \quad (11)$$

$$\dot{w}_0y_0+\sum_{i=0}^{13}\dot{a}_iy_{i+\frac{1}{2}}+\dot{c}hy_0^{'}+\dot{d}h^2y_0^{''}+\dot{e}h^3y_0^{'''}+\dot{u}h^4y_0^{(4)}+\dot{p}h^5y_0^{(5)}=h^{12}\sum_{i=0}^{15}\dot{b}_iy_{i+\frac{1}{2}}^{(12)} \quad (12)$$

$$\ddot{w}_ny_n+\sum_{i=0}^{13}\ddot{a}_iy_{i+n-\frac{25}{2}}+\ddot{c}hy_n^{'}+\ddot{d}h^2y_n^{''}+\ddot{e}h^3y_n^{'''}+\ddot{u}h^4y_n^{(4)}+\ddot{u}h^5y_n^{(5)}=h^{12}\sum_{i=0}^{15}\ddot{b}_iy_{i+n-\frac{29}{2}}^{(10)} \quad (13)$$

$$\dddot{w}_ny_n+\sum_{i=0}^{12}\dddot{a}_iy_{i+n-\frac{23}{2}}+\dddot{c}hy_n^{'}+\dddot{d}h^2y_n^{''}+\dddot{e}h^3y_n^{'''}+\dddot{u}h^4y_n^{(4)}+\dddot{p}h^5y_n^{(5)}=h^{12}\sum_{j=0}^{14}\dddot{b}_iy_{i+n-\frac{27}{2}}^{(12)} \quad (14)$$

$$w_0^{*}y_n+\sum_{i=0}^{11}a_i^{*}y_{i+n-\frac{21}{2}}+c^{*}hy_n^{'}+d^{*}h^2y_n^{''}+e^{*}h^3y_n^{'''}+u^{*}h^4y_n^{(4)}+p^{*}h^5y_n^{(5)}=h^{12}\sum_{j=0}^{13}b_i^{*}y_{i+n-\frac{25}{2}}^{(12)} \quad (15)$$

$$\overline{w}_0y_n+\sum_{i=0}^{10}\overline{a}_iy_{i+n-\frac{19}{2}}+\overline{c}hy_n^{'}+\overline{d}h^2y_n^{''}+\overline{e}h^3y_n^{'''}+\overline{u}h^4y_n^{(4)}+\overline{p}h^5y_n^{(5)}=h^{12}\sum_{i=0}^{12}\overline{b}_iy_{i+n-\frac{23}{2}}^{(12)} \quad (16)$$

$$\breve{w}_0y_n+\sum_{i=0}^{9}\breve{a}_iy_{i+n-\frac{17}{2}}+\breve{c}hy_n^{'}+\breve{d}h^2y_n^{''}+\breve{e}h^3y_n^{'''}+\breve{u}h^4y_n^{(4)}+\breve{p}h^5y_n^{(5)}=h^{12}\sum_{i=0}^{11}\breve{b}_iy_{i+n-\frac{21}{2}}^{(12)} \quad (17)$$

$$\tilde{w}_0y_n+\sum_{i=0}^{8}\tilde{a}_iy_{i+n-\frac{15}{2}}+\tilde{c}hy_n^{'}+\tilde{d}h^2y_n^{''}+\tilde{e}h^3y_n^{'''}+\tilde{u}h^4y_n^{(4)}+\tilde{u}h^5y_n^{(5)}=h^{12}\sum_{i=0}^{10}\tilde{b}_iy_{i+n-\frac{19}{2}}^{(12)} \quad (18)$$

Where all of the coefficients are arbitrary parameters to be determined.

4 Numerical results

In this section the presented method are applied to the following test problems if choosing

$$\alpha=\frac{-691}{130767436800},\beta=\frac{311}{9906624000},\gamma=\frac{11108959}{217945728000},\delta=\frac{250027007}{65383718400},$$

$$\eta=\frac{1604131663}{29059430400},\rho=\frac{443084947}{18162144000},\tau=\frac{6132449119}{15567552000}$$

We obtained the method of order $O(h^{26})$ respectively.

Example 1. We Consider the following boundary-value problem

$$y^{(12)}(x)+xy(x)=-(120+23x+x^3)e^x \;,\; 0\le x\le 1\;,$$
$$y(0)=0\;,y(1)=0,$$
$$y^{'}(0)=1, y^{'}(1)=-e,$$
$$y^{''}(0)=0, y^{''}(1)=-4e,$$

$y'''(0) = -3$, $y'''(1) = -9e$,

$y^{(4)}(0) = -8, y^{(4)}(1) = -16e$.

$y^{(5)}(0) = -15, y^{(5)}(1) = -25e$. (15)

The analytic solution of the above system is $y(x) = x(1-x)e^x$. It is evident from Table 1 that the maximum errors in absolute values are less than those presented by [8].

Example 2. We Consider the following boundary-value problem

$y^{(12)}(x) - y(x) = -12(2x\cos x + 11\sin x)$, $-1 \le x \le 1$,

$y(-1) = y(1) = 0,\quad y'(-1) = y'(1) = 2\sin(1)$,

$y''(-1) = -y''(1) = -4\cos(1) - 2\sin(1)$,

$y'''(-1) = y'''(1) = 6\cos(1) - 6\sin(1)$,

$y^{(4)}(-1) = -y^{(4)}(1) = 8\cos(1) + 12\sin(1)$,

$y^{(5)}(-1) = y^{(5)}(-1) = -20\cos(1) + 10\sin(1)$.

(16)

The analytic solution of the above system is $y(x) = (x^2 - 1)Sinx$. It is evident from Table 2 that the maximum errors in absolute values are less than those presented by [8].

Table1: Observed maximum absolute errors for example 1.

h	Our methods	[8]
$1/9$	1.44(−13)	7.38(−9)
$1/18$	7.18(−14)	-
$1/27$	8.71(−15)	-
$1/36$	4.09(−16)	-

Table2: Observed maximum absolute errors for example 2.

h	Our methods	[8]
$1/16$	5.67(−12)	1.14(−7)
$1/32$	8.01(−12)	-
$1/48$	9.11(−13)	-
$1/64$	5.84(−14)	-

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Conclusion

We approximate solution of the twelfth - order linear boundary-value problems by using non-polynomial spline. The new methods enable us to approximate the solution at every point of the range of integration. Tables 1-2 show that our methods produced better in the sense that max | e_i | is minimum in comparison with the methods developed in [8].

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