Numerical solution for pricing Asian option by using Block-Pulse functions

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Abstract: The valuation of path dependent Asian contingent claims is a difficult matter in mathematical finance. Only in some simple cases the no-arbitrage price of a path dependent contingent claim is computed in closed form. The numerical methods for solving arising equations are limited. In this paper, we propose a new method based on Block- Pulse Functions (BPFs), their operational matrix and direct method. Furthermore, we obtain an estimation of the error bound for this method by projection operator and prove the method is convergent. Numerical examples demonstrate the efficiency and accuracy of this approach.

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1. Introduction

Asian options are a variety of so-called exotic financial derivatives, where the contract specifies a future payoff depending on the average of stock price or index over a specified period in the future. These options can be more useful than ordinary or vanilla options, particularly in circumstances when an investor is more interested in average or regular exposure to an asset over a period rather than exposure on a particular data. There are many varieties for Asian option, for example: Fixed strike Asian options whose payoff is the difference between the average price and a fixed strike price, Floating strike options whose payoff is the difference between the final stock price and the average stock price and American versions of the Asian option, which allow for early exercise (opposed to the European version which can only be exercise at expiry). There are also variations in terms of how the averaging is defined. The most important is whether the average is arithmetic or geometric [4,5,8,14] . In this paper, we discussed arithmetic average option for European version. The pricing of Arithmetic Asian option has been tackled by a variety of analytical approximations and numerical algorithms. They are at least four methods for solving this problem. Monte Carlo simulations, which calculate the price by directly simulating the stock price process, numerical solution of a partial differential equation formulation of the problem, via finite difference or finite element methods [1,9,11], analytical representations in terms of infinite series and integral formulae, for example Laplace transform, which usually require numerical algorithms in order to recover the price[9], Density approximations which replace the sum of Lognormal density by a more tractable density[8], Lattices and Binomial trees which are related to finite difference method[13].

The PDE for an Asian call option with value $V(S,I,T)$ is:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} - rV = 0, \qquad (1)$$

with final condition

$$V(S,I,T) = \max\left(\frac{I}{T} - K, 0\right),$$

where,

$$I(t) = \int_0^t S(x)dx, \qquad (2)$$

is the average of the history of the asset price, where we know that this is independent of the current price. Also, we can treat S, I and T as independent variables.

This paper discusses the issue that arises in the valuation of these instruments in a no-arbitrage Black-Scholes $(B-S)$ frame work, as well how these problems may be solved via piecewise polynomials, for example Block Pulse Functions (BPFs), Haar functions or Walsh functions, etc. We solve (1) by two dimensional Block-Pulse functions and apply their operational matrix. Thus, the corresponding PDE is converted to a nonsingular linear system. The outline of this paper is as follows: In section 2, we obtain Analytic solution for (1). Block-Pulse functions are discussed in section 3. In section 4, error estimate for Block Pulse Functions are given. In section 5, operational matrix for partial derivatives are given. We use direct method for solving (1) in section 5. In section 6, error analysis for this method, using projection operator, is given. Numerical examples are given in section 7.

2. Analytical solution

In order to be able to solve the problem (1) subject to (2) numerically, we perform a variable transformation [13]:

$$x = \frac{Tk - tI}{S}, V(S, I, t) = \frac{S}{T} u(t, x), \quad (3)$$

by which Rogers and Shi [10] have reduced the PDE from two variables to one. By straight forward calculation, we have

$$\frac{\partial V}{\partial t} = \frac{S}{T}\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left(-\frac{I}{S}\right)\right),$$

$$\frac{\partial V}{\partial I} = \frac{S}{T}\left(\frac{\partial u}{\partial x}\left(-\frac{t}{S}\right)\right),$$

$$\frac{\partial V}{\partial I} = \frac{U}{T} + \frac{S}{T}\frac{\partial u}{\partial x}\left(\frac{tI - Tk}{S^2}\right),$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{2}{T}\frac{\partial u}{\partial x}\left(\frac{tI - Tk}{S^2}\right) + \frac{S}{T}\left(\frac{tI - Tk}{S^2}\right)^2 - \frac{2S}{T}\frac{\partial u}{\partial x}\left(\frac{tI - Tk}{S^3}\right), \quad (4)$$

substituting these into (1), we get

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 u}{\partial x^2} - (1 + rx)\frac{\partial u}{\partial x} = 0, \quad (5)$$

with final condition

$$u(x, T) = \max(-x, 0). \quad (6)$$

Thus, under the transformation (4), the arithmetic average of Asian option with fixed strike price is reduced to a Cauchy problem (4), (5) as a $1 - D$ parabolic equation in the domain $x \in \Re, 0 \le t \le T$. The PDE is defined on the whole real axis. Note that in [12,13], a formula was obtained for the case $x \le 0$ as:

$$V(S, I, t) = S\left(\frac{1 - e^{-r(T-t)}}{rT}\right) + e^{-r(T-t)}\left(\frac{I}{T} - k\right). \quad (7)$$

By making the change of variables as in (4), we get

$$u(x, t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) - xe^{-r(T-t)}. \quad (8)$$

We consider the solution of the PDE only for $x \ge 0$ using [8] for the boundary condition at $x = 0$, the complete system of Rogers and Shi's PDE is therefore

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 u}{\partial x^2} - (1 + rx)\frac{\partial u}{\partial x} = 0, x \ge 0, t \ge 0, \quad (9)$$

$$u(x, T) = 0, \quad (10)$$

$$u(0, t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right), \quad (11)$$

$$u(L, t) = 0. \quad (12)$$

Two dimensional Block-Pulse functions

Block-pulse functions have been studied and applied extensively as a basic set of functions for signal characterization in system science and control. This set of functions was first introduced to electrical engineering by Harmuth in 1969 [2,3]. A set of two dimensional Block-Pulse functions $\Phi_{i_1,i_2}(x,t)(i_1 = 0,1,2,\ldots,m_1 - 1, i_2 = 0,1,2,\ldots,m_2)$ is defined in the region $x \in [a, b)$ and $t \in [c, d)$ as:

$$\Phi_{i_1,i_2}(x,t) = \begin{cases} 1 & (i_1)h_1 \le x < (i_1 + 1)h_1, (i_2)h_2 \le t < (i_2 + 1)h_2 \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

where m_1, m_2 are arbitrary positive integers, and $h_1 = \frac{b-a}{m_1}, h_2 = \frac{d-c}{m_2}$. There are some properties for 2DBPFs as following:

The 2DBPFs are disjoint and orthogonal with each other. It is clear that

$$\int_c^d \int_a^b \phi_{i_1,i_2}(x,t)\phi_{j_1,j_2}(x,t)dxdt = \begin{cases} h_1 h_2 & i_1 = j_1, i_2 = j_2 \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

The 2DBPFs set is complete, when m_1 and m_2 approach infinity [6,7]. We can also expand a two variable function $f(x, t)$ into BPFs series:

$$f(x, t) \cong \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} f_{i_1,i_2} \Phi_{i_1,i_2}(x, t), \quad (15)$$

through determining the block pulse coefficients:

$$f_{i_1,i_2} = \frac{1}{h_1 h_2} \int_{(i_1)h_1}^{(i_1+1)h_1} \int_{(i_2)h_2}^{(i_2+1)h_2} f(x, t)dxdt, \quad (16)$$

in the region $x_1 \in [a, b)$ and $x_2 \in [c, d)$. Also, Parseval's identity holds:

$$\int_a^b \int_c^d (f(x,t))^2 dtdx = \sum_{i=0}^{m} \sum_{j=0}^{m} f_{i,j}^2 \left\|\phi_{i,j}\right\|^2 \quad (17)$$

Also, for vector forms, consider the m^2 elements of 2DBPFs

$$\Phi(x, t) = [\phi_{0,0}, \phi_{0,1}, \ldots, \phi_{0,m-1}, \ldots, \phi_{m-1,0}, \ldots, \phi_{m-1,m-1}]^T(x, t). \quad (18)$$

Above representation and disjointness property, follows:

$$\Phi(x,t)\Phi^T(x,t) = \begin{pmatrix} \phi_{0,0} & & & & 0 \\ & \phi_{0,1} & & & \\ & & \ddots & & \\ & & & \phi_{0,m-1} & \\ 0 & & & & \ddots \\ & & & & & \phi_{m-1,m-1} \end{pmatrix} \quad (19)$$

$$\int_c^d \int_a^b \Phi(x,t)\Phi^T(x,t)dxdt = h_1 h_2 I, \quad (20)$$

also,

$$\Phi^T(x,t)\Phi(x,t) = 1, \quad (21)$$

$$\Phi(x,t)\Phi^T(x,t)V = \tilde{V}\Phi(x,t), \quad (22)$$

where V is an m^2 vector and $\tilde{V} = diag(V)$. Moreover, it can be clearly concluded that for every $m^2 \times m^2$ matrix B:

$$\Phi^T(x,t)B\Phi(x,t) = \hat{B}^T\Phi(x,t), \quad (23)$$

where $\hat{B}$ is an m^2 column vector with elements equal to the diagonal entries of matrix B. For simplicity, from now on, we use $m_1 = m_2 = m$.

Operational matrix for partial derivatives

The expansion of a function $u(x,t)$ over $[0,1) \times [0,1)$ with respect to $\phi_{ij}(x,t), i,j = 0,1,\dots m-1$, can be written as

$$u(x,t) \cong \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} u_{i,j}\phi_{i,j}(x,t) = U\Phi^T = \Phi U^T, \quad (24)$$

where

$U = [u_{00}, u_{01}, \dots, u_{0m-1}, u_{10}, \dots, u_{1m-1}, \dots, u_{m-1,m-1}]$, $\Phi = [\phi_{0,0}, \phi_{0,1}, \dots, \phi_{0,m-1}, \phi_{1,0}, \dots, \phi_{1,m-1}, \dots, \phi_{m-1,m-1}]$, and

$$\phi_{ij}(x,t) = \begin{cases} 1 & \frac{i}{m} \le x < \frac{i+1}{m}, \frac{j}{m} \le t < \frac{j+1}{m} \\ 0 & \text{otherwise,} \end{cases} \quad (25)$$

$$u_{ij} = \frac{1}{h^2}\int_{\frac{i}{m}}^{\frac{i+1}{m}}\int_{\frac{i}{m}}^{\frac{i+1}{m}} u(x,t)dxdt. \quad (26)$$

Now, expressing $\int_0^1 \int_0^t \phi_{i,j}(s,y)dsdy$, in terms of the 2DBPFs as :

$$\int_0^1 \int_0^t \phi_{i,j}(s,y)dsdy \cong [0,0,\dots,0,\frac{h^2}{2},h^2,\dots,h^2], \quad (27)$$

in which $\frac{h^2}{2}$, is ith component. Thus

$$\int_0^1 \int_0^t \Phi(s,y)dsdy \cong P\Phi(x,t), \quad (28)$$

where P is $m^2 \times m^2$ matrix and is called operational matrix of double integration and can be denoted by $P = \frac{h^2}{2}P_2$, where

$$P_2 = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (29)$$

so, the double integral of every function $u(x,t)$ can be approximated by:

$$\int_0^1 \int_0^t u(s,y)dsdy \cong \frac{h^2}{2}U^T P_2 \Phi(x,t), \quad (30)$$

by similar method $\int_0^1 \Phi_{ij}(s,t)ds$, in terms of 2DBPFs as:

$$\int_0^1 \phi_{ij}(s,t)ds \cong [0,0,\dots,h,0,0,\dots,0]^T\Phi(.,.), \quad (31)$$

and

$$\int_0^1 \Phi(s,t)ds \cong hI\Phi(.,.). \quad (32)$$

Now, we compute operational matrix for $\frac{\partial u}{\partial t}$ as:

$$\int_0^1 \int_0^t \frac{\partial u(s,y)}{\partial y}dsdy \cong \frac{h^2}{2}(U_t^d)^T P_2 \Phi(x,t), \quad (33)$$

and

$$\begin{aligned} \int_0^1 \int_0^t \frac{\partial u(s,y)}{\partial y}dsdy &= \int_0^1 (u(s,t) - u(s,0))ds \\ &= \int_0^1 (U^T\Phi(s,t) - U_f^T\Phi(s,0))ds \quad (34) \\ &= hU^T I\Phi(x,t) - hU_f^T\Delta_1\Phi(x,t), \end{aligned}$$

where U_f^T is boundary vector and Δ_1 is the following $m^2 \times m^2$ matrix :

$$\Delta_1 = \begin{pmatrix} H_{m\times m} & & 0 \\ & H_{m\times m} & \\ 0 & \ddots & \\ & & H_{m\times m} \end{pmatrix}, \quad (35)$$

with

$$H_{m\times m} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (36)$$

from (31) and (35) we can conclude:

$$U_t^d \cong \frac{2}{h}(U^T - U_f^T\Delta_1)P_2^{-1}, \quad (37)$$

by the same method, operational matrix for $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ are given

$$\frac{\partial u}{\partial x} \cong (U_x^d)^T \Phi(x,t), \quad (38)$$

$$\frac{\partial^2 u}{\partial x^2} \cong (U_{xx}^d)^T \Phi(x,t), \quad (39)$$

where

$$U_x^d = \frac{1}{h}(U_{g_2}^T \Delta_3 - U_{g_1}^T \Delta_2)P_2^{-1}, \quad (40)$$

$$U_{xx}^d = \frac{1}{h^2}(U_{g_2}^T \Delta_3 - U_{g_1}^T \Delta_2)P_2^{-1}(\Delta_3 - \Delta_2)P_2^{-1}, \quad (41)$$

and Δ_2, Δ_3 are the following $m^2 \times m^2$ matrices:

$$\Delta_2 = \begin{pmatrix} I_{m\times m} & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}, \quad (42)$$

$$\Delta_3 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & I_{m\times m} \end{pmatrix}, \quad (43)$$

and U_{g_1}, U_{g_2} are boundary vectors of $u(0,t)$ and $u(1,t)$, respectively.

Direct method for solving nonlinear PDEs

The results obtained in previous section are used to introduce a direct efficient and simple method to solve equations $(10)-(13)$. In generality we consider equations $(10)-(13)$ of the form:

$$\frac{\partial u}{\partial t} = \frac{1}{2}a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x}, \quad (44)$$

$$u(x,0) = f(x),$$
$$u(0,t) = g_1(x),$$
$$u(1,t) = g_2(x). \quad (45)$$

Approximating functions $a(x,t)$ and $b(x,t)$ with respect to 2DBPFs we have:

$$a(x,t) \cong \Phi^T(x,t)A = A^T\Phi(x,t),$$
$$b(x,t) \cong \Phi^T(x,t)B = B^T\Phi(x,t), \quad (46)$$

hence

$$a(x,t)\frac{\partial^2 u}{\partial x^2} \cong \Phi^T(x,t)A(U_{xx}^d)^T\Phi(x,t)$$
$$= (A\widehat{(U_{xx}^d)^T})\Phi(x,t), \quad (47)$$

$$b(x,t)\frac{\partial u}{\partial x} \cong \Phi^T(x,t)B(U_x^d)^T\Phi(x,t)$$
$$= (B\widehat{(U_x^d)^T})\Phi(x,t). \quad (48)$$

By substituting the above equations into (45) and using boundary and initial conditions, we obtain a linear system with $u_{i,j}\,(i,j = 0,1,\dots,m-1)$ as unknowns:

$$(U_t^d)^T - \frac{1}{2}(\widehat{U_{xx}^d)^T}) + B\widehat{(U_x^d)^T} = 0. \quad (49)$$

Error analysis

Let the problem be of the form

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t) + f(x,t), \quad x \in [0,1), t \in [0,T)$$
$$u(x,0) = f(x), \quad (50)$$
$$u(0,t) = g_1(t),$$
$$u(1,t) = g_2(t),$$

where $f(x)$, $g_1(t)$, $g_1(t)$ belong to $L^2[0,1)$, and L is linear operator of the form

$$Lu = -\frac{\sigma^2}{2}x^2\frac{\partial^2 u(x,t)}{\partial x^2} + (1+rx)\frac{\partial u}{\partial x}, \quad (51)$$

It is assumed, $u(x,t)$ is an element of a Hilbert space $L^2[0,1) \times [0,T)$ with inner product $<.,.>$ and norm $\|.\|$ are bounded as follows:

$$< u(x,t), v(x,t) > = \int_0^T \int_0^1 u(x,t)v(x,t)dxdt, \quad (52)$$

$$\|u(x,t)\| = (\int_0^T \int_0^1 u^2(x,t)dxdt)^{\frac{1}{2}}. \quad (53)$$

Let P_m be the projection operator defined on $L^2[0,1) \times [0,T) \to \mathbb{B}$, where $\mathbb{B}$ is finite m^2 –dimensional, as:

$$u_m(x,t) = P_m u(x,t) = \sum_{j=0}^{m-1}\sum_{i=0}^{m-1} u_{i,j}\phi_{i,j}(x,t). \quad (54)$$

The discrete approximation of (51) is :

$$\frac{\partial u_m(x,t)}{\partial t} = L_m u_m(x,t) + f_m(x,t), \quad (55)$$

where, for each x,t, $u_m(x,t)$ belongs to an m^2 – dimensional subspace $\mathbb{B}$ and L_m is a linear operator form $L^2[0,1) \times [0,T)$ to $\mathbb{B}$ of the form $L_m = P_m L P_m$. (56)

First, we find an estimation of $\|u - P_m u\|$ for arbitrary $u \in L^2[0,1) \times [0,T)$.

Lemma : Let $u(x,t)$ be defined on $L^2[0,1) \times [0,T)$ and P_m be projection operator defined by (55) then $\|u - P_m u\| \le \frac{\sup|u|}{2\sqrt{3}m}$, (57)

where $up|u| = \max_{0\le i,j\le m-1}|u_{ij}|$ for $0 \le i,j \le m-1$.

Proof: The integral $\int_0^t \int_0^1 u_{i,j}\Phi(x,y)dxdy$ is a ramp $\frac{u_{i,j}}{m}(t - \frac{i}{m})$ on the subinterval $[\frac{i}{m}, \frac{i+1}{m}) \times [\frac{j}{m}, \frac{j+1}{m})$ with average value $\frac{u_{i,j}}{2m^2}$.

The error in approximating the ramp by this constant value over the subinterval $[\frac{i}{m}, \frac{i+1}{m}) \times [\frac{j}{m}, \frac{j+1}{m}) = I_{ij}$ is

$$r_{i,j}(s,t) = \frac{u_{i,j}}{2m^2} - \frac{u_{i,j}}{m}\left(t - \frac{i-1}{m}\right), \qquad (58)$$

hence, using E_{ij} as least square of the error on I_{ij}, we have

$$E_{i,j}^2 = \int_{\frac{j-1}{m}}^{\frac{j}{m}} \int_{\frac{i-1}{m}}^{\frac{i}{m}} (r_{i,j}(s,t))^2 ds dt \le \frac{|u_{i,j}|^2}{12m^6}, \qquad (59)$$

$$E_{i,j} \le \frac{|u_{i,j}|}{2\sqrt{3}m^3}, \qquad (60)$$

and on the interval $[0,1) \times [0,T)$ we have

$$\|u - P_m u\| = \max E_{i,j} \le \frac{\sup|u|}{2\sqrt{3}m}. \qquad (61)$$

Theorem: Let $u(x,t)$, $f(x,t)$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ be in $L^2[0,1) \times [0,T)$ and $u_m(x,t)$ be approximate solution by 2DBPFs and L be a linear operator defined as (52) such that

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t) + f(x,t),$$
$$\frac{\partial u_m(x,t)}{\partial t} = L_m u_m(x,t) + f_m(x,t) + e, \qquad (62)$$

then

$$\|e\| \le \frac{1}{2\sqrt{3}m}\left(\sup\left|\frac{\partial u}{\partial t}\right| + \sup|Lu| + \|(L^*)^2\|^{1/2}(\sup(u^2))^{1/2} + \sup|f|\right) \qquad (63)$$

where L^* is adjoint of L.

proof: By using properties of projection operators, $P_m^2 = P_m$ and $\|P_m\| = 1$ thus

$$e = \frac{\partial u_m(x,t)}{\partial t} - \frac{\partial u(x,t)}{\partial t} + Lu(x,t) - L_m u_m(x,t) + f(x,t) - f_m(x,t), \qquad (64)$$

$$= \frac{\partial(u_m - u)}{\partial t} + Lu - P_m L P_m P_m u + f - P_m f, \qquad (65)$$
$$= \frac{\partial(P_m u - u)}{\partial t} + Lu - P_m L P_m P_m u + f - P_m f,$$
$$= \frac{\partial(P_m - I)u}{\partial t} + Lu - P_m L P_m u + (I - P_m)f,$$
$$= \frac{\partial(P_m - I)u}{\partial t} + Lu - P_m Lu + P_m Lu - P_m L P_m u + (I - P_m)f,$$

$$= \frac{\partial(P_m - I)u}{\partial t} + (I - P_m)Lu + P_m L(I - P_m)u + (I - P_m)f,$$

$$\|e\| \le \left\|\frac{\partial(P_m - I)u}{\partial t}\right\| + \|(I - P_m)Lu\| + \|P_m L(I - P_m)u\| + \|(I - P_m)f\|, \qquad (66)$$

now, applying the above lemma we have

$$\|e\| \le \frac{\sup\left|\frac{\partial u}{\partial t}\right|}{2\sqrt{3}m} + \frac{\sup|Lu|}{2\sqrt{3}m} + \|P_m\|\|L(I - P_m)u\| + \frac{\sup|f|}{2\sqrt{3}m}, \qquad (67)$$

It is sufficient to fined a bound for $PL(I - P_m)uP, L(I - P_m)u = \sum_{j=0}^{m-1}\sum_{i=0}^{m-1} u_{i,j}\phi_{i,j}(x,t)$

where $u_{i,j}$ is: $u_{i,j} = \langle \phi_{i,j}, L(I - P_m)u \rangle = \langle L^*\phi_{i,j}, (I - P_m)u \rangle$,

$$\|L(I - P_m)u\| = \sum_{j=0}^{m-1}\sum_{i=0}^{m-1} |u_{i,j}| = \sum_{j=0}^{m-1}\sum_{i=0}^{m-1} |\langle L^*\phi_{i,j}, (I - P_m)u \rangle|,$$

using Cauchy-Schwartz inequality

$$\|L(I - P_m)\| \le \|(L^*)^2\|^{1/2}\|(I - P_m)u^2\|^{1/2},$$

by substituting in (67)

$$\|e\| \le \frac{1}{2\sqrt{3}m}\left(\sup\left|\frac{\partial u}{\partial t}\right| + \sup|Lu| + \|(L^*)^2\|^{1/2}(\sup(u^2))^{1/2} + \sup|f|\right), \qquad (68)$$

$$\|e\| \le \frac{A}{2\sqrt{3}m}, \qquad (69)$$

where

$A = \sup\left|\frac{\partial u}{\partial t}\right| + \sup|Lu| + \|(L^*)^2\|^{1/2}(\sup(u^2))^{1/2} + \sup|f|)$

for $(x,t) \in [0,1) \times [0,T)$, so by hypothesis of the theorem, A is a finite number and $\|e\| = O(\frac{1}{m})$. So, if $m \to \infty$ then $\|e\|$ tends to zero.

Numerical examples

Consider the following two examples. We solve them by direct method and numerical results obtained here can be compared with exact solution $u(x,0)$. The numerical results show that with increasing m, the approximate solution gets better. To show the accuracy of the method we report infinity norm of the error which is defined by

$$\|e\| = \|u(x,0) - u_m(x,0)\|_\infty. \qquad (70)$$

Example 1

In $(8)-(11)$ let $T = r = \sigma = L = 1$, numerical results for different m are shown in Table 1. Error function between exact solution and numerical solution is plotted in Figure 1 for $m = 5$.

Table 1: Error between exact solution and numerical solution in U(x,0).

m	5	10	20	30
e	1.0945e-4	2.8247e-4	3.7763e-5	1.1421e-5

Example 2

In $(8)-(11)$ let $T=1$, $r=0.1$, $\sigma=0.2$, $L=1$. Numerical results for some m are given in Table 2. Error function between exact solution and numerical solution are plotted in Figure 2 for $M=5$.

Table 2: Error between exact solution and numerical solution in U(x,0).

m	5	10	20	30
e	2.7934e-4	3.8351e-4	5.2384e-5	1.5950e-5

Figure 1: error function for u(x, 0) for example 1, m=5.

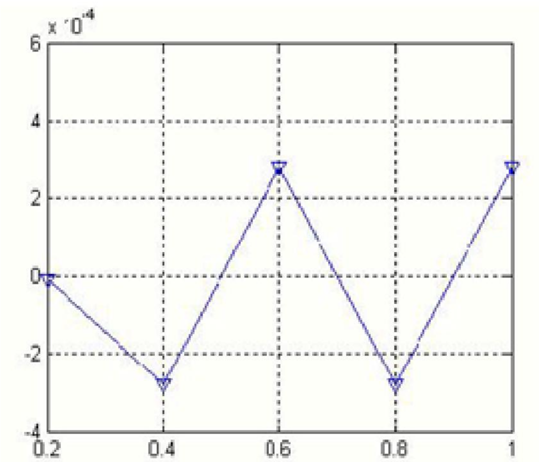
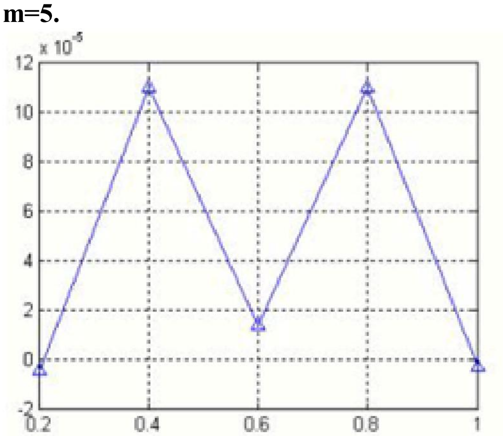


Figure 2: error function for u(x, 0) for example 2, m=5.



Conclusion

In this paper, we introduced a new method for solving partial differential equations. We used 2DBPFs and its operational matrix to solve Asian option problem. The proposed method is simple theoretically, thus we can use it for solving linear and nonlinear partial differential equations. Also, we can apply this method to other complex options, for example, American option, Exotic option, etc.

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