Fuzzy AG-Subgroups

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Abstract. An AG-group is a generalization of an abelian group. A groupoid (G, $\cdot$) is called an AG-group, if it satisfies the identity $(ab)c = (cb)a$, called the left invertive law, contains a unique left identity and inverse of its every element. We extend the concept of AG-group to fuzzy AG-group. We define and investigate some structural properties of fuzzy AG-subgroup.

[I. Ahmad, Amanullah, and M. Shah. **Fuzzy AG-Subgroups.** *Life Sci J* 2012;9(4):3931-3936]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 586

Keywords: AG-group; conjugate AG-group; normal AG-group, fuzzy AG-subgroup

1. Introduction

A fuzzy subset μ of a set X is a function from X to the unit closed interval [0, 1]. The concept of fuzzy subset of a set was initiated by Lofti A. Zadeh [1] in 1965. Zadeh's ideas developed new directions for researchers worldwide. The concept of fuzzy subset of a set has a lot of applications in various fields like computer engineering, AI, control engineering, operation research, management sciences and many more [7]. Lots of researches in this field show its importance and applications in set theory, algebra, real analysis, measure theory and topological spaces [2]. Rapid theoretical development's and practical applications based on the concept of a fuzzy subset in various fields are in progress.

In 1971, Azriel Rosenfeld introduced the notion of a fuzzy subgroup [4]. Recently the structure of AG-groupoid (a groupoid satisfying the left invertive law: $(ab)c = (cb)a$ is fuzzified [10]. An AG-groupoid is a generalization of a commutative semigroup. It is also easy to verify that an AG-groupoid always satisfies the medial law: $(ab)(cd) = (ac)(bd)$. Many features of AG-groupoids can be studied in [6], newly discovered classes of AG-groupoids and their enumeration has been done in [11], Quasi-cancellativity of AG-groupoids can be seen in [12] and construction of some algebraic structures from AG-groupoids and vice versa can be found in [13]. In the present paper we are fuzzifying the structure of AG-group initiated by [10]. An AG-group is related to an AG-groupoid as a group to a semigroup. An AG-group is one of the most interesting non-associative structures. There is no commutativity and associativity in AG-group in general, but an AG-group becomes an abelian group if any one of these holds in it. AG-groups are not power associative otherwise it becomes an abelian group. For these and further studies on AG-groups we refer the reader to [5, 6].

2. Preliminaries

In this section we list some basic definitions that will frequently be used in the subsequent sections of this paper.

A fuzzy subset of X is a function from X into the unit closed interval [0, 1]. The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by $\mathrm{FP}(X)$. Let $\mu \in \mathrm{FP}(X)$. Then the set $\{\mu(x) : x \in X\}$ is called the image of μ and is denoted by $\mu(X)$ or $\mathrm{Im}(X)$. The set $\{x : x \in X, \mu(x) > 0\}$, is called support of μ and is denoted by μ^*. In particular, μ is called a finite fuzzy subset if μ^* is a finite set, and an infinite fuzzy subset otherwise. Let $\mu, \nu \in \mathrm{FP}(X)$. If $\mu(x) \le \nu(x)$ for all $x \in X$, then μ is said to be contained in ν (or ν contains μ), and we write $\mu \subseteq \nu$ (or $\nu \supseteq \mu$). If $\mu \subseteq \nu$ and $\mu \ne \nu$, then μ is said to be properly contained in ν (or ν properly contains μ) and we write $\mu \subset \nu$ (or $\nu \supset \mu$). Let

$\mu, \nu \in \mathrm{FP}(X)$. Define $\mu \cup \nu$ and $\mu \cap \nu \in \mathrm{FP}(X)$ as follows: $\forall x \in X$,

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$$

$$(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x).$$

In this case $\mu \cup \nu$ and $\mu \cap \nu$ are called the union and the intersection of μ and ν respectively.

Lemma 1. [5, Lemma 1] Let G be an AG-group. Let $a,b,c,d \in G$ and e is the left identity in G. Then the following conditions hold in G:

(i) $(ab)(cd) = (db)(ca)$, paramedial law;

(ii) $a \cdot bc = b \cdot ac$;

(iii) $(ab)^{-1} = a^{-1}b^{-1}$;

(iv) $(ab)(cd) = (dc)(ba)$.

Remark 2. We will not reproduce a proof in the paper if it is similar to that of groups.

3. Fuzzy AG-subgroups

In the rest of this paper G will denote an AG-group unless otherwise stated and e will denote the left identity of G.

Definition 3. Let $\mu \in \mathrm{FP}(G)$, then μ is called a fuzzy AG-subgroup of G if for all $x, y \in G$;

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$;

(ii) $\mu(x^{-1}) \geq \mu(x)$.

We will denote the set of all fuzzy AG-subgroups of G by $\mathrm{F}(G)$. μ satisfies conditions (i) and (ii) of Definition 3, if and only if

$$\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)\ ;\ \forall x, y \in G.$$

Definition 4. If $\mu \in \mathrm{F}(G)$, then

$$\mu_* = \{x : x \in G, \mu(x) = \mu(e)\}, \text{ and}$$

$$\mu^* = \{x : x \in G, \mu(x) > 0\}$$

μ^* is called the support of G.

Proposition 5. If $\mu \in \mathrm{F}(G)$. Then μ_* is an AG-subgroup of G.

Proposition 6. If $\mu \in \mathrm{F}(G)$. Then μ^* is an AG-subgroup of G.

Definition 7. Let $\mu \in \mathrm{F}(G)$. For $\alpha \in [0,1]$, define μ_α as follows:

$$\mu_\alpha = \{x : x \in G, \mu(x) \geq \alpha\}$$

μ_α is called the α-cut, (or α-level set) of μ.

Definition 8. The binary operation "o" and unary operation "$^{-1}$" on $\mathrm{FP}(G)$ is defined as follows: for all $\mu, \nu \in \mathrm{FP}(G)$ and for all $x \in G$,

$$(\mu \,\mathrm{o}\, \nu)(x) = \vee\{\mu(y) \wedge \nu(z) : y, z \in G, yz = x\}$$

and

$$(\mu^{-1})(x) = \mu(x^{-1}).$$

We call $\mu \,\mathrm{o}\, \nu$ the product of μ and ν, and μ^{-1} the inverse of μ.

Lemma 9. [4, Lemma 1.2.5] Let $\mu \in \mathrm{F}(G)$. Then for all $x \in G$,

(i) $\mu(e) \geq \mu(x)$;

(ii) $\mu(x) = \mu(x^{-1})$.

Theorem 10. Let $\mu \in \mathrm{FP}(G)$. Then $\mu \in \mathrm{F}(G)$ if and only if μ satisfies the following conditions:

(i) $\mu \,\mathrm{o}\, \mu \subseteq \mu$;

(ii) $\mu^{-1} \subseteq \mu$ (or $\mu^{-1} \supseteq \mu$, or $\mu^{-1} = \mu$).

The following theorem holds for groups on the condition of commutativity, and also holds for AG-groups without commutativity.

Theorem 11. Let $\mu, \nu \in \mathrm{F}(G)$. Then $\mu \,\mathrm{o}\, \nu \in \mathrm{F}(G)$.

Proof. By [10, Proposition 1], $\mathrm{FP}(G)$ is an AG-groupoid. Now using Lemma 1-(iii) we have

$$\begin{aligned}(\mu \,\mathrm{o}\, \nu)^{-1} &= \mu^{-1} \,\mathrm{o}\, \nu^{-1} \\ &= \mu \,\mathrm{o}\, \nu \qquad \text{(by Theorem 10)}\end{aligned}$$

also we have,

$$\begin{aligned}(\mu \,\mathrm{o}\, \nu)\,\mathrm{o}\,(\mu \,\mathrm{o}\, \nu) &= (\mu \,\mathrm{o}\, \mu)(\nu \,\mathrm{o}\, \nu) \qquad \text{(by medial law)} \\ &\subseteq \mu \,\mathrm{o}\, \nu \qquad \text{(by Theorem 10)}\end{aligned}$$

Hence both the conditions of Theorem 10, are satisfied. Therefore, $\mu \,\mathrm{o}\, \nu \in \mathrm{F}(G)$. ■

Next we give an alternative proof of the same fact. Let $\mu, \nu \in \mathrm{F}(G)$. Then we have for all $x \in G$,

$$((\mu \,\mathrm{o}\, \nu)\,\mathrm{o}\,(\mu \,\mathrm{o}\, \nu))(x) = \vee\{(\mu \,\mathrm{o}\, \nu)(y) \wedge (\mu \,\mathrm{o}\, \nu)(z) : y, z \in G, x = yz\}$$

$$= \vee\{(\vee\{\mu(y_1) \wedge \nu(y_2) : y_1, y_2 \in G, y = y_1y_2\})$$
$$\wedge (\vee\{\mu(z_1) \wedge \nu(z_2) : z_1, z_2 \in G, z = z_1z_2\})\}$$
$$= \vee\{(\vee\{\mu(y_1) \wedge \mu(z_1) : y_1, z_1 \in G\})$$
$$\wedge (\vee\{\nu(y_2) \wedge \nu(z_2) : y_2, z_2 \in G : x = (y_1y_2)(z_1z_2)\}$$

$= \vee\{(\vee\{\mu(y_1)\wedge\mu(z_1): y_1, z_1 \in G\})\wedge$
$(\vee\{\nu(y_2)\wedge\nu(z_2): y_2, z_2 \in G: x=(y_1z_1)(y_2z_2)\}$ (by medial law)
$\subseteq \vee\{(\mu(x)\wedge\nu(x): x=(y_1z_1)(y_2z_2)\}$
$= (\mu \circ \nu)(x)$
$\Rightarrow ((\mu\circ\nu)\circ(\mu\circ\nu))(x) \subseteq (\mu\circ\nu)(x)$

Also we have

$(\mu\circ\nu)^{-1}(x) = \mu\circ\nu(x^{-1})$ (by Definition 8)
$= \vee\{\mu(y^{-1})\wedge\nu(z^{-1}): y,z\in G, x=yz \Rightarrow x^{-1}=(yz)^{-1}=y^{-1}z^{-1}\}$ (by Definition 8)
$= \vee\{\mu(y)\wedge\nu(z): y,z\in G, x=yz\}$ $(\mu,\nu\in \mathrm{F}(G))$
$= (\mu\circ\nu)(x)$ (by Definition 8)
$\Rightarrow (\mu\circ\nu)^{-1}(x) = (\mu\circ\nu)(x)\ \forall x\in G$
$\Rightarrow (\mu\circ\nu)^{-1} = (\mu\circ\nu)$

Hence both the conditions of Theorem 10, are satisfied. Therefore, $\mu\circ\nu \in \mathrm{F}(G)$. ■

Theorem 12. Let $\mu \in \mathrm{FP}(G)$. Then the following assertions are equivalent; for all $x, y \in G$,

(i) $\mu(xy)=\mu(yx)$;
(ii) $\mu(ye)=\mu(y)$;
(iii) $\mu(ye)\geq\mu(y)$;
(iv) $\mu(ye)\leq\mu(y)$.

Proof. (i) ⇒ (ii) : Let $y\in G$. Then

$\mu(ye)=\mu(y\cdot x^{-1}x)$
$=\mu(x^{-1}\cdot yx)$ (by Lemma 1-(ii))
$=\mu(yx\cdot x^{-1})$ (since $\mu(xy) = \mu(yx)$)
$=\mu(x^{-1}x\cdot y)$ (by left invertive law)
$=\mu(ey)$
$=\mu(y)$.

(ii) ⇒ (iii) : Obvious.

(iii) ⇒ (iv) : Let $y\in G$. Then

$\mu(ye)\leq\mu(ye\cdot e)$
$=\mu(ee\cdot y)$ (by left invertive law)
$=\mu(ey)$
$=\mu(y)$.

(iv) ⇒ (i) : Let $x,y\in G$. Then

$\mu(xy)=\mu(ex\cdot y)$
$=\mu(yx\cdot e)$ (by left invertive law)
$\leq\mu(yx)$
$=\mu(ey\cdot x)$
$=\mu(xy\cdot e)$ (by left invertive law)
$\leq\mu(xy)$.

Thus $\mu(xy)\leq\mu(yx)\leq\mu(xy)$. Hence $\mu(xy) = \mu(yx)$. ■

Corollary 13. Let $\mu\in\mathrm{F}(G)$. Then the following assertions holds; for all $x,y\in G$,

(i) $\mu(xy)=\mu(yx)$;
(ii) $\mu(ye)=\mu(y)$;
(iii) $\mu(ye)\geq\mu(y)$;
(iv) $\mu(ye)\leq\mu(y)$.

Proof. Since μ is an AG-subgroup. So μ always satisfies Condition (iii) of Theorem 12. As indeed,

$\mu(ye)\geq\mu(y)\wedge\mu(e)$
$=\mu(y)\quad\forall\quad y\in G.$

Hence by Theorem 12, all the conditions holds always. ■

Theorem 14. $(\mu\circ\nu)(x)=(\nu\circ\mu)(xe)\ \forall\mu,\nu\in \mathrm{FP}(G)$ and $x\in G$.

Proof. Let $x\in G$. Then we have

$(\mu\circ\nu)(x)=\vee_{y\in G}\{\mu(xy)\wedge\nu(y^{-1})\}$ (by Definition 8)
$=\vee_{y\in G}\{\nu(y^{-1})\wedge\mu(xy)\}$
$=(\nu\circ\mu)(y^{-1}\cdot xy)$ (by Definition 8)
$=(\nu\circ\mu)(x\cdot y^{-1}y)$ (by Lemma 1-(ii))
$=(\nu\circ\mu)(xe)$. ■

Corollary 15. $(G,\cdot)$ is commutative ⇔ $(\mathrm{FP}(G),\circ)$ is commutative.

Remark 16. We note that if μ is a fuzzy AG-subgroup of an AG-group G and if $x,y\in G$ with $\mu(x)\neq\mu(y)$, then $\mu(xy)=\mu(x)\wedge\mu(y)$.

Suppose $\mu(x)>\mu(y)$. Then

$\mu(y)=\mu(ey)$
$=\mu(x^{-1}x\cdot y)$
$=\mu(yx\cdot x^{-1})$ (by left invertive law)
$\geq\mu(yx)\wedge\mu(x^{-1})$
$=\mu(xy)\wedge\mu(x)$ (by Lemma 9, and Cor. 13).

Thus $\mu(y)\geq\mu(xy)\wedge\mu(x)$ and since $\mu(x)\geq\mu(y)$, it follows that

$\mu(y)\geq\mu(xy)$
$\geq\mu(x)\wedge\mu(y)$
$=\mu(y)$.

Thus $\mu(xy)=\mu(x)\wedge\mu(y)$. A similar argument can be used for the case $\mu(y) > \mu(x)$.

Lemma 17. Let μ be a fuzzy AG-subgroup of G. Let $x \in G$ then $\mu(xy) = \mu(y)$ $\forall y \in G$ if and only if $\mu(x) = \mu(e)$.

Proof. Suppose that $\mu(xy) = \mu(y)$ $\forall y \in G$, then by letting $y = e$, we get that

$$\mu(xe) = \mu(e)$$

$$\Rightarrow \mu(x) = \mu(e) \quad \text{(by Corollary 13)}$$

Conversely, assume that $\mu(x) = \mu(e)$. Then by Lemma 9,

$$\mu(x) = \mu(e) \geq \mu(y) \ \ \forall y \in G$$

$\Rightarrow \mu(x) \geq \mu(y)$ $\forall y \in G$, and so,

$\mu(xy) \geq \mu(x) \wedge \mu(y) = \mu(y)$. Also,

$$\begin{aligned}
\mu(y) &= \mu(ey) \\
&= \mu(x^{-1}x \cdot y) \\
&= \mu(yx \cdot x^{-1}) \quad \text{(by left invertive law)} \\
&\geq \mu(yx) \wedge \mu(x^{-1}) \\
&= \mu(xy) \wedge \mu(x) \quad \text{(by Lemma 9, and Cor. 13)} \\
&= \mu(xy)
\end{aligned}$$

Thus $\mu(xy) \geq \mu(y) \geq \mu(xy)$.

Hence $\mu(xy) = \mu(y)$ $\forall y \in G$. ■

Definition 18. If $\mu, \nu \in \mathrm{F}(G)$ and there exists $u \in G$ such that $\mu(x) = \nu(ux \cdot u^{-1})$ $\forall x \in G$, then μ and ν are called conjugate fuzzy AG-subgroups (with respect to u) and we write, $\mu = \nu^u$, where $\nu^u(x) = \nu(ux \cdot x^{-1})$ for all $x \in G$.

Definition 19. Let $\mu \in \mathrm{F}(G)$. Then μ is called a normal fuzzy AG-subgroup of G if,

$$\mu(xy \cdot x^{-1}) = \mu(y) \ \forall x, y \in G.$$

We will denote the set of all normal fuzzy AG-subgroups of G by $\mathrm{NF}(G)$.

Theorem 20. Let $\mu \in \mathrm{F}(G)$. Then the following assertions are equivalent; for all $x, y \in G$,

(i) $\mu(xy \cdot x^{-1}) = \mu(y)$;

(ii) $\mu(xy \cdot x^{-1}) \geq \mu(y)$;

(iii) $\mu(xy \cdot x^{-1}) \leq \mu(y)$.

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii)$\Rightarrow$ (iii):

$$\begin{aligned}
\mu(xy \cdot x^{-1}) &\leq \mu((x^{-1}(xy \cdot x^{-1})) \cdot (x^{-1})^{-1}) \\
&= \mu((x^{-1}(xy \cdot x^{-1})) \cdot x) \\
&= \mu((x(xy \cdot x^{-1})) \cdot x^{-1}) \quad \text{(by left invert. law)} \\
&= \mu(((xy)(xx^{-1})) \cdot x^{-1}) \quad \text{(by Lemma 1)} \\
&= \mu((xy \cdot e) \cdot x^{-1}) \\
&= \mu((ey \cdot x) \cdot x^{-1}) \quad \text{(by left invertive law)} \\
&= \mu(yx \cdot x^{-1}) \\
&= \mu(x^{-1}x \cdot y) \quad \text{(by left invertive law)} \\
&= \mu(ey) \\
&= \mu(y)
\end{aligned}$$

$\Rightarrow \mu(xy \cdot x^{-1}) \leq \mu(y)$ $\forall x, y \in G$

(iii) $\Rightarrow$ (i):

$$\begin{aligned}
\mu(xy \cdot x^{-1}) &\geq \mu((x^{-1}(xy \cdot x^{-1})) \cdot (x^{-1})^{-1}) \\
&= \mu(y), \text{ (as in the proof (ii) } \Rightarrow \text{ (iii))}
\end{aligned}$$

$\Rightarrow \mu(xy \cdot x^{-1}) \geq \mu(y)$ $\forall x, y \in G$

Hence $\mu(xy \cdot x^{-1}) = \mu(y)$. ■

Lemma 21. Let $\mu \in \mathrm{FP}(G)$. Then μ is a fuzzy AG-subgroup of G if and only if μ_a is an AG-subgroup of G, $\forall a \in \mu(G) \cup \{b \in [0,1]: b \leq \mu(e)\}$.

Theorem 22. Let $\mu \in \mathrm{FP}(G)$. Then $\mu \in \mathrm{NF}(G)$ if and only if μ_a is a normal AG-subgroup of G

$$\forall a \in \mu(G) \cup \{b \in [0,1]: b \leq \mu(e)\}.$$

Proof. Suppose that $\mu \in \mathrm{NF}(G)$. Let $a \in \mu(G) \cup \{b \in [0,1]: b \leq \mu(e)\}$. Since $\mu \in \mathrm{F}(G)$, μ_a is an AG -subgroup of G. If $x \in G$ and $y \in \mu_a$, it follows from Theorem 20, that

$\mu(xy \cdot x^{-1}) = \mu(y) \geq a \Rightarrow \mu(xy \cdot x^{-1}) \geq a$,

thus $xy \cdot x^{-1} \in \mu_a$. Hence μ_a is a normal AG-subgroup of G.

Conversely, assume that μ_a is a normal AG-subgroup of G $\forall a \in \mu(G) \cup \{b \in [0,1]: b \leq \mu(e)\}$. By Lemma 21, we have $\mu \in \mathrm{FP}(G)$. Let $x, y \in G$ and $a = \mu(y)$. Then $y \in \mu_a$ and so $xy \cdot x^{-1} \in \mu_a$.

Hence $\mu(xy \cdot x^{-1}) \geq a = \mu(y)$. That is, μ satisfies Condition (ii) of Theorem 20. Consequently, it follows from Theorem 20, that $\mu \in \mathrm{NF}(G)$. ■

Theorem 23. Let $\mu \in \mathrm{NF}(G)$. Then μ_* and μ^* are normal AG-subgroups of G.

Proof. Since $\mu \in \mathrm{F}(G)$, it follows from Propositions 5 and 6, that μ_* and μ^* are AG-subgroups of G. Let $x \in G$ and $y \in \mu_*$. Since μ satisfies Condition (i) of Theorem 20, we have $\mu(xy \cdot x^{-1}) = \mu(y) = \mu(e)$ and thus $xy \cdot x^{-1} \in \mu_*$. Hence μ_* is a normal AG-subgroups of G. Now let $x \in G$ and $y \in \mu^*$. Since μ satisfies Condition (i) of Theorem 20, it follows that $\mu(xy \cdot x^{-1}) = \mu(y) > 0 \Rightarrow \mu(xy \cdot x^{-1}) > 0$, thus $xy \cdot x^{-1} \in \mu^*$. Hence μ^* is normal AG-subgroups of G. ■

Theorem 24. Suppose $\mu \in \mathrm{FP}(G)$. Let $N(\mu) = \{x : x \in G, \mu(xy) = \mu(yx)\ \forall y \in G\}$. Then $N(\mu)$ is either empty or an AG-subgroup of G if the restriction of μ to $N(\mu)$, $\mu\big|_{N(\mu)}$ is a normal fuzzy AG-subgroup of $N(\mu)$.

Proof. Here we discuss two cases.

Case: 1. If also $\mu \in \mathrm{F}(G)$, then by Corollary 13, $N(\mu) = G$ and the theorem holds trivially.

Case: 2 Suppose $\mu \notin \mathrm{F}(G)$. Clearly $N(\mu)$ is nonempty, because $\mu(ey) = \mu(e)$ and $\mu(ye) = \mu(y) \Rightarrow \mu(ey) = \mu(ye)\ \forall y \in G \Rightarrow e \in N(\mu)$. Let $x, y \in N(\mu)$. For any $z \in G$, we see that

$$\begin{aligned} \mu(xy^{-1} \cdot z) &= \mu(zy^{-1} \cdot x) && \text{(by left invert. law)} \\ &= \mu(x \cdot zy^{-1}) && \text{(by Theorem 12)} \\ &= \mu(z \cdot xy^{-1}) && \text{(by Lemma 1-(ii))} \end{aligned}$$

Thus $xy^{-1} \in N(\mu)$. Hence $N(\mu)$ is an AG-subgroup of G. Now by [4, Comment 1.2.4] if $\mu \in \mathrm{F}(G)$ and H is a subgroup of G, then $\mu\big|_H \in \mathrm{F}(H)$, consequently $\mu\big|_{N(\mu)} \in \mathrm{F}(N(\mu))$. ■

The fuzzy AG-subgroup $N(\mu)$ of G defined in Theorem 24, is called the normalizer of μ in G.

Definition 25. For $x, y \in G$. Then the commutator $[x, y]$ of AG-group G is defined as

$$[x, y] = (xy)(y^{-1}x^{-1}).$$

Theorem 26. Let μ be a fuzzy AG-subgroup of G. Then $\mu \in \mathrm{NF}(G)$ if and only if

$$\mu([x, y]) \geq \mu(x)\ \ \forall x, y \in G.$$

Proof. Suppose μ is a normal fuzzy AG-subgroup of G. Let $x, y \in G$, then

$$\begin{aligned} \mu([x, y]) &= \mu((xy)(y^{-1}x^{-1})) \\ &= \mu((y^{-1}x^{-1})(xy)) && \text{(by Corollary 13)} \\ &= \mu((yx)(x^{-1}y^{-1})) && \text{(by Lemma 1-(iv))} \\ &= \mu(x^{-1} \cdot ((yx)y^{-1})) && \text{(by Lemma 1-(ii))} \\ &\geq \mu(x^{-1}) \wedge \mu(yx \cdot y^{-1}) \\ &\geq \mu(x) \wedge \mu(x) && \text{(by Lemma 9, and } \mu \in \mathrm{NF}(G)) \\ &= \mu(x). \end{aligned}$$

Hence $\mu([x, y]) \geq \mu(x)\ \ \forall x, y \in G$.

Conversely, assume that μ satisfies the given inequality. Then for all $x, y \in G$, we have

$$\begin{aligned} \mu(xz \cdot x^{-1}) &= \mu(e(xz \cdot x^{-1})) \\ &= \mu((zz^{-1})(xz \cdot x^{-1})) \\ &= \mu(((xz \cdot x^{-1})z^{-1})z) && \text{(by left invert. law)} \\ &= \mu(((z^{-1}x^{-1})(xz))z) && \text{(by left invert. law)} \\ &= \mu((xz)(x^{-1}z^{-1})z) && \text{(by Lemma 1-(iv))} \\ &= \mu([z, x]z) \\ &\geq \mu([z, x]) \wedge \mu(z) \\ &\geq \mu(z) \wedge \mu(z) = \mu(z). \end{aligned}$$

Thus $\mu(xz \cdot x^{-1}) \geq \mu(z)\ \forall x \in G$. Then by Theorem 20, we have $\mu(xz \cdot x^{-1}) = \mu(z)\ \forall x \in G$. Hence μ is normal fuzzy AG-subgroup of G. ■

Proposition 27. Let μ be a fuzzy AG-subgroup of G. Then $\mu([x, y]) = \mu(e)\ \ \forall x, y \in G$ if and only if μ is normal fuzzy AG-subgroup of G.

Proof. Suppose $\mu \in \mathrm{NF}(G)$. Then we have

$$\begin{aligned} & \mu(yx \cdot y^{-1}) = \mu(x)\ \forall x, y \in G \\ \Leftrightarrow\ & \mu(e(yx \cdot y^{-1})) = \mu(x) \\ \Leftrightarrow\ & \mu((xx^{-1})(yx \cdot y^{-1})) = \mu(x) \\ \Leftrightarrow\ & \mu(((yx \cdot y^{-1})x^{-1})x) = \mu(x) && \text{(by left invertive law)} \\ \Leftrightarrow\ & \mu(((x^{-1}y^{-1})(yx))x) = \mu(x) && \text{(by left invertive law)} \\ \Leftrightarrow\ & \mu(((xy)(y^{-1}x^{-1}))x) = \mu(x) && \text{(by Lemma 1-(iv))} \\ \Leftrightarrow\ & \mu([x, y] \cdot x) = \mu(x) \\ \Leftrightarrow\ & \mu([x, y]) = \mu(e)\text{; by lemma 17.} \end{aligned}$$

■

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11/19/2012