On a class of analytic functions defined by Ruscheweyh derivative

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Abstract: The aim of this paper is to introduce a class of analytic functions defined by using generalized janowski functions and Ruscheweyh derivative. The coefficient bound, inclusion result and a radius problem has been discussed in this paper. Several known results have been deducted from our main results as special cases by assigning particular values to the different parameters.
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1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ in E such that $f(z) = g(w(z))$.

Let $P[A,B]$ be the class of functions h, analytic in E with $h(0) = 1$ and

$$h(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1.$$

This class was introduced by Janowski [1]. The class $P[A,B]$ is connected with the class P of functions with positive real parts by the relation

$$h \in P[A,B] \Leftrightarrow \frac{(B-1)h-(A-1)}{(B+1)h-(A+1)} \in P. \qquad (1.2)$$

Later Polatoğlu [2] defined the class $P[A,B,\alpha]$ as:

Let $P[A,B,\alpha]$ be the class of functions p_1, analytic in E with $p_1(0) = 1$ and

$$p_1(z) \prec \frac{1+\{(1-\alpha)A+\alpha B\}z}{1+Bz}, \qquad (1.3)$$

where $-1 \le B < A \le 1$, $0 \le \alpha < 1$.

From (1.3), it can easily be seen that, $p_1 \in P[A,B,\alpha]$, if and only if, there exists $h \in P[A,B]$ such that

$$p_1(z) = (1-\alpha)h(z) + \alpha, \quad 0 \le \alpha < 1, \quad z \in E. \qquad (1.4)$$

It is also noted that $P[1,-1,0] \equiv P$, the well-known class of analytic functions in E with positive real part. Noor [3] considered the generalized class $P_k[A,B,\alpha]$ of Janowski functions which is defined as follows.

A function p is said to be in the class $P_k[A,B,\alpha]$, if and only if,

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \qquad (1.5)$$

where

$p_1, p_2 \in P[A,B,\alpha]$, $-1 \le B < A \le 1$, $k \ge 2$, and $0 \le \alpha < 1$. It is clear that $P_2[A,B,\alpha] \equiv P[A,B,\alpha]$ and $P_k[1,-1,0] \equiv P_k$, the well-known class given and studied by Pinchuk [4].

For any two analytic functions

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } f_2(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in E),$$

the convolution (Hadamard product) of f_1 and f_2 is defined by

$$\left(f_1 * f_2\right)(z)=\sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.6}$$

Using Hadamard product, Ruscheweyh [5] introduced a linear operator $D^\delta : A \to A.$ It is defined as

$$\begin{aligned} D^\delta f(z) &= \frac{z}{(1-z)^{\delta+1}} * f(z) \\ &= z+\sum_{n=2}^{\infty} \varphi_n(\delta)\ a_n\ z^n, \quad (\delta>-1) \end{aligned} \tag{1.7}$$

with

$$\varphi_n(\delta)=\frac{(\delta+1)_{n-1}}{(n-1)!}, \tag{1.8}$$

where $(\rho)_n$ is a Pochhammer symbol given as

$$(\rho)_n=\begin{cases} 1, & n=0, \\ \rho(\rho+1)(\rho+2)\ldots(\rho+n-1), & n\in \mathrm{N}. \end{cases} \tag{1.9}$$

Moreover, for $\delta \in \mathrm{N}_0=\{0,1,2,\ldots\}$,

$$D^\delta f(z)=\frac{z\left(z^{\delta-1} f(z)\right)^{(\delta)}}{\delta!}.$$

The function $D^\delta f(z)$ was then called δth order Ruscheweyh derivative of f. For the application of Ruscheweyh derivative, see [6–8]. The following identity can easily be established.

$$(\delta+1) D^{\delta+1} f(z)=\delta D^\delta f(z)+z\left(D^\delta f(z)\right)'. \tag{1.10}$$

Now using all these concepts, we define the following class.

Definition 1.1. A function $f\in A$ is in the class $V_k^\delta[A,B,\alpha,b]$, if and only if ,

$$\left(1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1} f(z)}{D^\delta f(z)}\right)\in P_k[A,B,\alpha],\quad z\in E,$$

where $k\geq 2,\ \delta>-1,\quad -1\leq B<A\leq 1,\ 0\leq\alpha<1$ and $b\in \mathrm{C}-\{0\}$.

Assigning certain values to different parameters, we have different well-known classes of analytic functions as can be seen below.

Special cases

(i) $V_k^\delta[1,-1,\alpha,b]\equiv V_k(\alpha,b,\delta)$, the well-known class defined by Latha and Nanjunda Rao in [9].

(ii) $V_2^1[A,B,\alpha,1]\equiv C[A,B,\alpha]$, $V_2^0[A,B,\alpha,2]\equiv S^*[A,B,\alpha]$, the well-known class defined by Polatoğlu [2].

(iii) $V_k^1[A,B,0,1]\equiv V_k[A,B]$, $V_2^0[A,B,0,2]\equiv R_k[A,B]$, where $V_k[A,B]$ and $R_k[A,B]$ denote the class of janowski functions with bounded boundary and bounded radius rotations respectively, given by Noor [10].

For the detail on the subject of functions of bounded boundary rotation, Janowski functions and related topics, we refer the work of Noor et.al [11] and Arif et.al [12].

2. Preliminary Results

We need the following results to obtain our main results.

Lemma 2.1. Let $p(z)=1+\sum_{n=1}^{\infty} q_n z^n\in P_k[A,B,\alpha]$. Then, for all $n\geq 1$,

$$|q_n|\leq \frac{k(A-B)(1-\alpha)}{2}. \tag{2.1}$$

This inequality is sharp.

The proof follows from (1.4), (1.5) and the coefficient bound of $h\in P[A,B]$ given by Aouf [13].

Lemma 2.2 [14]. Let $u=u_1+iu_2$, $v=v_1+iv_2$ and $\psi(u,v)$ be a complex valued function satisfying the conditions:

(i) $\psi(u,v)$ is continuous in a domain $D\subset \mathbb{C}^2$,

(ii) $(1,0)\in D$ and $\mathrm{Re}\,\psi(1,0)>0$,

(iii) $\mathrm{Re}\,\psi(iu_2,v_1)\leq 0$, whenever $(iu_2,v_1)\in D$ and $v_1\leq -\frac{1}{2}\left(1+u_2^2\right)$.

If $h(z)=1+c_1z+\cdots$ is a function analytic in E such that $(h(z),zh'(z))\in D$ and $\mathrm{Re}\,\psi(h(z),zh'(z))>0$ for $z\in E$, then $\mathrm{Re}\,h(z)>0$ in E.

Lemma 2.3. Let $p\in P_k[A,B,0]$ with $k\geq 2$.

Then, for $|z| = r < 1$,

$$\frac{2-(A-B)kr-2ABr^2}{2(1-B^2r^2)} \le \operatorname{Re} p(z) \le |p(z)| \le \frac{2+(A-B)kr-2ABr^2}{2(1-B^2r^2)}. \tag{2.2}$$

The proof is immediate by using (1.5) and the growth result of $h \in P[A,B]$, see [15].

Lemma 2.4. Let $p \in P_k[A,B,0]$ with $k \ge 2$. Then, for $|z| = r < 1$,

$$|zp'(z)| \le \frac{r\left\{\begin{array}{l}(A-B)k-4B(A-B)r\\ +B^2(A-B)kr^2\end{array}\right\}\operatorname{Re} p(z)}{(1-B^2r^2)(2+(A-B)kr-2ABr^2)}. \tag{2.3}$$

The result follows directly by using Lemma 2.3.

3. Main Results

Theorem 3.1. Let $f \in V_k^{\delta}[A,B,\alpha,b]$ with $-1 \le B < A \le 1$, $\delta > -1$, $b \in \mathrm{C}-\{0\}$, $0 \le \alpha < 1$. Then

$$|a_n| \le \frac{(\sigma)_{n-1}}{(n-1)!\varphi_n(\delta)}, \quad \forall\, n \ge 2, \tag{3.1}$$

where $\sigma = \dfrac{k|b|(A-B)(1-\alpha)(\delta+1)}{4}$ and $\varphi_n(\delta)$ is given by (1.8).

This result is sharp.

Proof. Set

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} = p(z), \tag{3.2}$$

so that $p \in P_k[A,B,\alpha]$. Let $p(z) = 1+\sum_{n=1}^{\infty} q_n z^n$. Then (3.2) can be written as

$$2\left(D^{\delta+1}f(z)-D^{\delta}f(z)\right) = bD^{\delta}f(z)\sum_{n=1}^{\infty} q_n z^n,$$

which implies that

$$\frac{2\varphi_n(\delta)(n-1)a_n}{(\delta+1)} = b\begin{pmatrix} q_{n-1}+\varphi_2(\delta)\ a_2\ q_{n-2}+ \\ \cdots+\varphi_{n-1}(\delta)\ a_{n-1}\ q_1\end{pmatrix}.$$

Using Lemma 2.1 , we obtain

$$|a_n| \le \frac{k|b|(A-B)(1-\alpha)(\delta+1)}{4(n-1)\varphi_n(\delta)}\begin{pmatrix}1+\varphi_2(\delta)\ |a_2|+\cdots \\ +\varphi_{n-1}(\delta)\ |a_{n-1}|\end{pmatrix}$$
$$= \frac{\sigma}{(n-1)\varphi_n(\delta)}\left(1+\sum_{i=2}^{n-1}\varphi_i(\delta)\ |a_i|\right).$$

For $n = 2$, $|a_2| \le \frac{\sigma}{\varphi_2(\delta)} = \frac{(\sigma)2-1}{(2-1)!\varphi_2(\delta)}$.

Therefore (3.1) holds for $n = 2$.

Assume that (3.1) is true for $n = m$ and consider

$$|a_{m+1}| \le \frac{\sigma}{m\,\varphi_{m+1}(\delta)}\left(1+\sum_{i=2}^{m}\varphi_i(\delta)\ |a_i|\right)$$

$$\le \frac{\sigma}{m\,\varphi_{m+1}(\delta)}\left(1+\sum_{i=2}^{m}\frac{(\sigma)_{i-1}}{(i-1)!}\right)$$

$$= \frac{\sigma}{m\,\varphi_{m+1}(\delta)}\left(1+\sum_{i=2}^{m}\sigma\prod_{j=1}^{i-1}\left(1+\frac{\sigma}{j}\right)\right)$$

$$= \frac{\sigma}{m\,\varphi_{m+1}(\delta)}\prod_{j=1}^{m-1}\left(1+\frac{\sigma}{j}\right)$$

$$= \frac{(\sigma)_m}{(m)!\,\varphi_{m+1}(\delta)}.$$

Therefore, the result is true for $n = m+1$. Using mathematical induction, (3.1) holds true for all $n \ge 2$.

This result is sharp for $\delta > -1$, $0 \le \alpha < 1$, $b \in \mathrm{C}-\{0\}$ and $k \ge 2$ as can be seen from the functions $f_0(z)$ which are given as

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f_0(z)}{D^{\delta}f_0(z)} = (1-\alpha)\begin{bmatrix}\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+Az}{1+Bz} \\ -\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-Az}{1-Bz}\end{bmatrix}+\alpha.$$

For different values of A, B, α, b and δ, we obtain the following results [16].

Corollary 3.2. If $f \in V_k^0[1,-1,\alpha,2] = R_k(\alpha)$, then

$$\left|a_n\right| \le \frac{\left(k\left(1-\alpha\right)\right)_{n-1}}{\left(n-1\right)!}, \quad \forall\, n \ge 2.$$

This result is sharp.

Corollary 3.3. If $f \in V_k^1\left[1,-1,\alpha,1\right] = V_k\left(\alpha\right)$, then

$$\left|a_n\right| \le \frac{\left(k\left(1-\alpha\right)\right)_{n-1}}{n!}, \quad \forall\, n \ge 2.$$

This result is sharp.

Theorem 3.4. For real $b > 0$,

$$V_k^{\delta+1}\left[A,B,\alpha,b\right] \subseteq V_k^{\delta}\left[1,-1,\beta,b+1\right], \quad z \in E,$$

where $\beta\left(0 \le \beta < 1\right)$ is one of the roots of

$$\begin{aligned}&\lambda_1\lambda_2 b^2\left(\delta+2\right)^2\left(1-\alpha\right)^2\\ &-b\left(\delta+2\right)\left(1-\alpha\right)\left[\lambda_1\left(B+1\right)+\lambda_2\left(B-1\right)\right]\\ &\qquad+\left(B^2-1\right)=0,\end{aligned} \tag{3.3}$$

where

$$\lambda_1 = \frac{\left(1-b\right)+\beta\left(1+b\right)}{\left(1+b\right)\left(1-\beta\right)}\left[\Delta\left(B-1\right)-\left(A-1\right)\right], \tag{3.4}$$

$$\lambda_2 = \frac{\left(1-b\right)+\beta\left(1+b\right)}{\left(1+b\right)\left(1-\beta\right)}\left[\Delta\left(B+1\right)-\left(A+1\right)\right] \tag{3.5}$$

and

$$\Delta = 1-\frac{\left(1+b\right)\left(\delta+1\right)\left(1-\beta\right)}{b\left(\delta+2\right)\left(1-\alpha\right)}.$$

Proof. Suppose $f \in V_k^{\delta+1}\left[A,B,\alpha,b\right]$ and set

$$p\left(z\right) = 1-\frac{2}{b+1}+\frac{2}{b+1}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}. \tag{3.6}$$

where p is analytic in E with $p\left(0\right)=1$. Then, by simple computations together with (3.6) and (1.10) yield

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} = \left(1-\mu_1\right)+\mu_1\left[p(z)+\frac{\mu_2\, zp'\left(z\right)}{p\left(z\right)+\mu_3}\right], \tag{3.7}$$

where $\mu_1 = \frac{\delta+1}{\delta+2}\frac{b+1}{b}$, $\mu_2 = \frac{2}{(\delta+1)(b+1)}$, $\mu_3 = \frac{2}{b+1}-1$.

Since $f \in V_k^{\delta+1}\left[A,B,\alpha,b\right]$, it follows that

$$\left(1-\mu_1\right)+\mu_1\left[p(z)+\frac{\mu_2\, zp'\left(z\right)}{p\left(z\right)+\mu_3}\right] \in P_k\left[A,B,\alpha\right],$$

or, equivalently

$$\frac{\left(1-\alpha-\mu_1\right)}{1-\alpha}+\frac{\mu_1}{1-\alpha}\left[p(z)+\frac{\mu_2\, zp'\left(z\right)}{p\left(z\right)+\mu_3}\right] \in P_k\left[A,B\right]. \tag{3.8}$$

Define

$$\varphi\left(z\right) = \frac{1}{\left(1+\mu_3\right)}\frac{z}{\left(1-z\right)^{\mu_2}}+\frac{\mu_3}{\left(1+\mu_3\right)}\frac{z}{\left(1-z\right)^{\mu_2+1}},$$

and by using convolution techniques given by Noor [3], we have

$$\begin{aligned}p(z)+\frac{\mu_2\, zp'\left(z\right)}{p\left(z\right)+\mu_3} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1\left(z\right)+\frac{\mu_2\, zp_1'\left(z\right)}{p_1\left(z\right)+\mu_3}\right)\\ &\quad-\left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2\left(z\right)+\frac{\mu_2\, zp_2'\left(z\right)}{p_2\left(z\right)+\mu_3}\right).\end{aligned}$$

By using (3.8), we see that

$$\frac{\left(1-\alpha-\mu_1\right)}{1-\alpha}+\frac{\mu_1}{1-\alpha}\left[p_i\left(z\right)+\frac{\mu_2\, zp_i'\left(z\right)}{p_i\left(z\right)+\mu_3}\right] \in P\left[A,B\right],$$

where $z \in E$, $i = 1,2$.

Now, we want to show that $p_i \in P\left[A,B,\beta\right]$, where $\beta\left(0 \le \beta < 1\right)$ is one of the root of (3.3).

Let

$$p_i\left(z\right) = \left(1-\beta\right)h_i\left(z\right)+\beta, \quad i = 1,2.$$

Then

$$\frac{1-\alpha-\mu_1\left(1-\beta\right)}{1-\alpha}+\frac{\mu_1\left(1-\beta\right)}{1-\alpha}\left[h_i\left(z\right)+\frac{\frac{\mu_2}{(1-\beta)}\, zh_i'\left(z\right)}{h_i\left(z\right)+\frac{\mu_3+\beta}{(1-\beta)}}\right] \in P\left[A,B\right].$$

Using the fact illustrated in (1.2), we have

$$\left\{\frac{(B-1)\begin{bmatrix}(\lambda+\mu h_i(z))(h_i(z)+\omega_2)\\ +\omega_1\mu z h_i'(z)\end{bmatrix} -(A-1)(h_i(z)+\omega_2)}{(B+1)\begin{bmatrix}(\lambda+\mu h_i(z))(h_i(z)+\omega_2)\\ +\omega_1\mu z h_i'(z)\end{bmatrix} -(A+1)(h_i(z)+\omega_2)}\right\} \in P,$$

where $\omega_1 = \frac{\mu_2}{1-\beta}$, $\omega_2 = \frac{\mu_3+\beta}{1-\beta}$, $\lambda = \frac{1-\alpha-\mu_1(1-\beta)}{1-\alpha}$ and $\mu = \frac{\mu_1(1-\beta)}{1-\alpha}$. We now form the functional $\psi(u,v)$ by choosing $u = h_i(z)$, $v = zh_i'(z)$ and note that the first two conditions of Lemma 2.2 are clearly satisfied. We check condition (iii) as follows.

$$\psi(u,v) = \frac{\left\{\begin{matrix}(B-1)\left[(\lambda+\mu u)(u+\omega_2)+\omega_1\mu v\right]\\ -(A-1)(u+\omega_2)\end{matrix}\right\}}{\left\{\begin{matrix}(B+1)\left[(\lambda+\mu u)(u+\omega_2)+\omega_1\mu v\right]\\ -(A+1)(u+\omega_2)\end{matrix}\right\}}$$

$$= \frac{\left\{\begin{matrix}\lambda_1+\omega_1\mu(B-1)v\\ +\left[(\lambda+\mu(u+\omega_2))(B-1)-(A-1)\right]u\end{matrix}\right\}}{\left\{\begin{matrix}\lambda_2+\omega_1\mu(B+1)v\\ +\left[(\lambda+\mu(u+\omega_2))(B+1)-(A+1)\right]u\end{matrix}\right\}},$$

where $\lambda_1 = \omega_2\left[\lambda(B-1)-(A-1)\right]$ and $\lambda_2 = \omega_2\left[\lambda(B+1)-(A+1)\right]$. Now

$$\psi(iu_2,v_1) = \frac{\left\{\begin{matrix}\lambda_1+\mu\left(\omega_1 v_1-u_2^2\right)(B-1)\\ +\left[(\lambda+\mu\omega_2)(B-1)-(A-1)\right]iu_2\end{matrix}\right\}}{\left\{\begin{matrix}\lambda_2+\mu\left(\omega_1 v_1-u_2^2\right)(B+1)\\ +\left[(\lambda+\mu\omega_2)(B+1)-(A+1)\right]iu_2\end{matrix}\right\}}.$$

Taking real part of $\psi(iu_2,v_1)$, we have

$$\operatorname{Re}\psi(iu_2,v_1) = \frac{\begin{matrix}\left[-\lambda_1+\mu\left(\omega_1 v_1-u_2^2\right)(1-B)\right]\left[\lambda_2+\mu\left(\omega_1 v_1-u_2^2\right)(B+1)\right]-\\ \left[(\lambda+\mu\omega_2)(B-1)-(A-1)\right]\left[(\lambda+\mu\omega_2)(B+1)-(A+1)\right]u_2^2\end{matrix}}{-\left[\lambda_2+\mu\left(\omega_1 v_1+u_2\right)(B+1)\right]^2-\left[(\lambda+\mu\omega_2)(B+1)-(A+1)\right]^2 u_2^2}.$$

As $\omega_1 > 0$, $\mu > 0$, so applying $v_1 \le -\frac{1}{2}\left(1+u_2^2\right)$ and after a little simplification, we have

$$\operatorname{Re}\psi(iu_2,v_1) \le \frac{A_1+B_1u_2^2+C_1u_2^4}{D_1}, \qquad (3.9)$$

where

$$A_1 = \frac{1}{4}\left[2\lambda_1-\omega_1\mu(B-1)\right]\left[2\lambda_2-\omega_1\mu(B+1)\right],$$

$$B_1 = -\frac{1}{2}\mu(\omega_1+2)\begin{bmatrix}\lambda_1(B+1)-\omega_1\mu\left(B^2-1\right)\\ +\lambda_2(B-1)\end{bmatrix}+$$
$$(\lambda+\mu\omega_2)^2\left(B^2-1\right)-2(\lambda+\mu\omega_2)(AB-1)$$
$$+\left(A^2-1\right),$$

$$C_1 = -\frac{1}{4}\mu^2\left(1-B^2\right)(\omega_1+2)^2,$$

and

$$D_1 = \left[\lambda_2+\mu\left(\omega_1v_1+u_2\right)(B+1)\right]^2$$
$$+\left[(\lambda+\mu\omega_2)(B+1)-(A+1)\right]^2 u_2^2.$$

The right hand side of (3.9) is negative if $A_1 \le 0$ and $B_1 \le 0$. From $A_1 \le 0$, we have β to be one of the roots of

$$\lambda_1\lambda_2 b^2(\delta+2)^2(1-\alpha)^2 -$$
$$b(\delta+2)(1-\alpha)\left[\lambda_1(B+1)+\lambda_2(B-1)\right]$$
$$+\left(B^2-1\right)=0$$

with $0 \le \beta < 1$ and also for $0 \le \beta < 1$, we have $B_1 \le 0$.

Since all the conditions of Lemma 2.2 are satisfied, it follows that $h_i \in P$, $i = 1,2$ and consequently $p \in P_k\left[1,-1,\beta\right]$. Hence from (3.6),

$$f \in V_k^{\delta}\left[1,-1,\beta,b+1\right].$$

By choosing the parameters $A = 1$, $B = -1$, $b = 1$ and $\delta = 0$, we obtain the following known result, proved in [17].

Corollary 3.5. Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where β is a root of

$$2\beta^2-(2\alpha-1)\beta-1=0 \text{ with } 0 \le \beta < 1,$$

which is

$$\beta=\frac{1}{4}\left[(2\alpha-1)+\sqrt{4\alpha^2-4\alpha+9}\right].$$

For $\alpha=0$, $k=2$ in Corollary 3.5, we have the following well known result [18].

$$V_2(0)=C\subseteq R_2\left(\frac{1}{2}\right)=S^*\left(\frac{1}{2}\right), \text{ for } z\in E.$$

Theorem 3.6. Let $f\in V_k^{\delta}[A,B,0,b]$, $\delta>-1$, $b>0$ (real), $k\geq 2$ and $0<a=\frac{b(\delta+1)}{2}\leq 1$. Then $D^{\delta}f(z)$ maps $|z|<r_0$ onto a convex domain, where r_0 is the least positive root of the equation

$$a_1r^4+a_2r^3+a_3r^2+a_4r+4(2a-1)=0 \text{ with } 0\leq r<1, \tag{3.10}$$

where

$$a_1=4a^2A^2B^2-4(a-1)^2B^4,$$
$$a_2=2a(2a-1)(B-A)B^2k,$$
$$a_3=8a^2(a-2)+8a(1-a)AB-a^2(A-B)^2k^2,$$

and

$$a_4=2a(2a-3)(A-B)k.$$

This result is sharp.

Proof. Since $f\in V_k^{\delta}[A,B,0,b]$, then

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}=\frac{b(p(z)-1)+2}{2}, \tag{3.11}$$

where $p\in P_k[A,B,0]$. Using the identity (1.10), we have from (3.11),

$$\frac{z\left(D^{\delta}f(z)\right)'}{D^{\delta}f(z)}=\frac{b(p(z)-1)(\delta+1)+2}{2}. \tag{3.12}$$

Logarithmic differentiation of (3.12) yields

$$\frac{\left(z\left(D^{\delta}f(z)\right)'\right)'}{\left(D^{\delta}f(z)\right)'}=ap(z)-a+1+\frac{zp'(z)}{p(z)-1+\frac{1}{a}},$$

where $a=\frac{b(\delta+1)}{2}$. Then, we have

$$\mathrm{Re}\left(1+\frac{z\left(D^{\delta}f(z)\right)''}{\left(D^{\delta}f(z)\right)'}\right)\geq a\,\mathrm{Re}\,p(z)+(1-a)$$
$$-\frac{|zp'(z)|}{\left|p(z)-1+\frac{1}{a}\right|},$$

and hence, by using Lemma 2.3 and Lemma 2.4 ,

$$\mathrm{Re}\left(1+\frac{z\left(D^{\delta}f(z)\right)''}{\left(D^{\delta}f(z)\right)'}\right)$$

$$\geq \mathrm{Re}\,p(z)\left\{a+\frac{2(1-a)(1-B^2r^2)}{2+(A-B)kr-2ABr^2}\right.$$

$$\left.-\frac{2ar\left\{(A-B)k-4B(A-B)r+B^2(A-B)kr^2\right\}}{\left(2+(A-B)kr-2ABr^2\right)\xi}\right\}$$

$$=\mathrm{Re}\,p(z)\left\{\frac{a_1r^4+a_2r^3+a_3r^2+a_4r+4(2a-1)}{\left(2+(A-B)kr-2ABr^2\right)\xi}\right\}>0,$$

provided

$$T(r)=a_1r^4+a_2r^3+a_3r^2+a_4r+4(2a-1)>0,$$

where

$$a_1=4a^2A^2B^2-4(a-1)^2B^4,$$
$$a_2=2a(2a-1)(B-A)B^2k,$$
$$a_3=8a^2(a-2)+8a(1-a)AB-a^2(A-B)^2k^2,$$
$$a_4=2a(2a-3)(A-B)k,$$

and

$$\xi=2(2a-1)-a(A-B)kr+2\left(B^2-a(A+B)B\right)r^2.$$

We have $T(0)>0$ and $T(1)<0$. Therefore, $D^{\delta}f(z)$ maps $|z|<r_0$ onto a convex domain, where r_0 is the least positive root of the equation $T(r)=0$, lying in $(0,1)$.

For $D^{\delta}f_1(z)$ such that

$$\frac{D^{\delta+1}f_1(z)}{D^{\delta}f_1(z)}=\frac{b(p_k(z)-1)+2}{2},$$

where $p_k(z)=\frac{2+(A-B)kz-2ABz^2}{2(1-B^2z^2)}$, we have

$$\frac{\left(z\left(D^{\delta}f_1(z)\right)'\right)'}{\left(D^{\delta}f_1(z)\right)'}=\frac{a_1r^4+a_2r^3+a_3r^2+a_4r+4\left(2a-1\right)}{\left(2+\left(A-B\right)kr-2ABr^2\right)\xi}$$
$$=0,$$

for $z=r_0$. Hence this radius r_0 is sharp.

By choosing the parameters $A=1$, $B=-1$, $k=2$, $b=2$ and $\delta=0$, we obtain the following known result, see [18].

Corollary 3.7. Let $f\in S^*$. Then f maps $|z|<r_0$ onto a convex domain, where r_0 is the least positive root of the equation

$$r^4-2r^3-6r^2-2r+1=0 \text{ with } 0\le r<1,$$

which is $r_0=2-\sqrt{3}$. This is also sharp.

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