Optimal Homotopy Asymptotic Method for the Approximate Solution of Generalized Burgers' Huxley Equation

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Abstract: In this paper, the approximate analytical solution of Generalized Burgers'-Huxley equations is obtained using Optimal Homotopy Asymptotic Method. Unlike homotopy perturbation and homotopy analysis methods this method is independent of the small parameter. Using this method one can easily handle the convergence of approximation series and adjustment of convergence regions when required. The method is effective, explicit and easy to implement. Approximate solution of Generalized Burgers'-Huxley equation, and its special cases Burgers'-Huxley equation and Huxley equation are considered using the present approach. The results show excellent accuracy and strength of the proposed method.

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INTRODUCTION

Nonlinear partial differential equations arise in distinct fields of engineering and science such as chemistry, physics, engineering and finance, and are the key point for the mathematical formulation of continuum models [1, 5, 19-20]. Particularly, the important model of nonlinear partial differential equation is Generalized Burgers'-Huxley equation [3, 5-6, 11, 13-14, 19, 22, 29]. The Burger's–Huxley equation was first studied by Satsuma [24] in 1987, and is given by,

$$u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma),\ x\in\Omega=[a,b],\ t\ge 0, \quad (1)$$

with initial condition,

$$u(x,0) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\omega_1 x)\right)^{\frac{1}{\delta}},\ x\in\Omega,$$

and boundary conditions,

$$u(x,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\omega_1 (x-\omega_2 t))\right)^{\frac{1}{\delta}},\ x\in\Gamma, t>0 \quad (2)$$

where $\omega_1 = \dfrac{-\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma$,

$$\omega_2 = \frac{\alpha\gamma}{(1+\delta)} - \frac{(1+\delta-\gamma)\left(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)}\right)}{2(1+\delta)}$$ and

α, β, δ and γ are constants so that $\beta \ge 0$, $\delta > 0$, $\gamma \in (0,1)$ and Γ is boundary of Ω. Eq. (1) is the combine form of Burgers and Huxley equations. When $\alpha = 0$, $\delta = 1$, Eq.(1) reduces into the form of Huxley equation. Huxley equation is used in nerve pulse propagation during nerve fibers and wall motion in liquid crystals [23, 25-28]. When $\beta = 0$, Eq.(1) gives us a well known form of the Burgers' equation [3, 8, 30], which has a certain role in shock wave model, sound waves in viscous medium, boundary layer characteristics and traffic flow, and its detail study was presented by Hon and Mao [8] using multi quadric radial basis function. Several methods for the solution of Generalized Burgers'-Huxley equation have been introduced in the literature including variational iteration method [3], spectral collocation method [5], adomian decomposition method [6], meshless method [14] and finite difference method [22]. In this paper, we use a recently developed method for the solution of Generalized Burgers'-Huxley equation, which is known as Optimal Homotopy Asymptotic Method [17]. This method has been successfully used for the solution of various ordinary and partial differential equations (see [2, 9-10, 12, 17-18, 21]). The main advantage of this method over other perturbation methods is that it is independent of the small parameter. This small parameter plays a very important role in determining the accuracy of the other perturbation methods as well as their validity. The exertion of the small parameter into the equation is the difficulty of perturbation methods. Therefore, it is the small parameter that greatly restricts the application of the perturbation method. Furthermore, the homotopy perturbation method and homotopy analysis method are special cases of the optimal homotopy asymptotic method.

In the next section, we develop the proposed method for the solution of Generalized Burgers'-Huxley equation.

METHOD OF OHAM

In this section, we present the optimal homotopy asymptotic method for the solution of boundary value problem of the form

$$T\left(u(x,t)\right)+f(x,t)=0,\ x\in\Omega,\ t\geq 0, \qquad (2)$$

$$B\left(u(x,t),\frac{\partial u(x,t)}{\partial t}\right)=0,\ x\in\Gamma, \qquad (3)$$

where T is a differential operator, B is boundary operator, $u(x,t)$ is the solution of problem (2)-(3), x and t are spatial and temporal independent variables, Γ is the boundary of the Ω and $f(x,t)$ is a known analytic function. Now according to the basic formulation of OHAM, T can be split into two differential operators, say L and N such that

$$L\left(u(x,t)\right)+N\left(u(x,t)\right)+f(x,t)=0,\ x\in\Omega,$$

where L is the differential (linear) operator of Eq.(2) so that it is the simplest part of the differential Eq. (2) and its analytical solution is easily available. N is the differential (nonlinear) operator of Eq. (2) so that it is the complicated (remaining) part of differential Eq.(2), whose analytical solution may or may not be easily available. Let (assuming that) $u_0(x,t):\Omega\to R$ is the solution of

$$L\left(u_0(x,t)\right)+f(x,t)=0,\ B\left(u_0(x,t),\frac{\partial u_0(x,t)}{\partial t}\right)=0, \qquad (4)$$

and is continuous function. $u(x,t):\Omega\to R$ is the solution of Eq.(2), which is also continuous. Then according to OHAM, we can define a homotopy $F\left(x,t;p\right):\Omega\times[0,1]\to R$ which satisfies

$$(1-p)\left\{L\left(F(x,t;p)\right)+f(x,t)\right\}-H(p)\left\{T\left(F(x,t;p)\right)+f(x,t)\right\}=0, \qquad (5)$$

where $x\in\Omega$ and $p\in[0,1]$ is the embedding parameter, $H(p)$ is a auxiliary function for Eq.(2) such that $H(p)\neq 0$ for all $p\in(0,1)$ and $H(0)=0$. Obviously, we have at $p=0$, Eq.(5) becomes $L\left(u_0(x,t)\right)+f(x,t)=0$ and at $p=1$, Eq. (5) becomes

$$N\left(u(x,t)\right)+L\left(u(x,t)\right)+f(x,t)=0.$$

Also clearly by definition of homotopy

$$F(x,t;p)=u_0(x,t),\ \text{at}\ p=0,$$

$$F(x,t;p)=u(x,t),\ \text{at}\ p=1.$$

Thus, we have as p varies from 0 to 1, $F(x,t;p)$ varies (or deforms) from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is the solution of problem given in Eq.(4) which is obtained from Eq.(5) and Eq.(3) at $p=0$. Now according to OHAM, one can choose general form of the auxiliary function $H(p)$ for the differential equation such as

$$H(p)=C_1p+C_2p^2+C_3p^3+........+C_kp^k+.... \qquad (3)$$

where $C_1,C_2,C_3,...,C_k,...$ are constants to be determined later. (4)

We want to approach to approximate solution of (2). For this, we expand $F(x,t;p,C_1,C_2...)$ in Taylor's series with respect to p as

$$F(x,t;p,C_1,C_2,...)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t;C_1,C_2...C_k)p^k. \qquad (6)$$

According to the value of $F(x,t;p,C_1,C_2...)$ in Eq. (6), we expand Eq.(5) and equating the coefficients of like powers of p. Then in addition, we obtained the zeroth-order problem defined in Eq. (4) and the obtained first and second order problems are defined by

$$L(u_1(x,t))=C_1N_0(u_0(x,t)),$$

$$B\left(u_1(x,t),\frac{\partial u_1(x,t)}{\partial t}\right)=0, \quad \text{and}$$

$$L(u_2(x,t))=C_2N_0(u_0(x,t))+C_1N_1\left(u_0(x,t),u_1(x,t)\right)+(1+C_1)L\left(u_1(x,t)\right), \qquad (5)$$

$$B\left(u_2(x,t),\frac{\partial u_2(x,t)}{\partial t}\right)=0, \text{ respectively.}$$

In general, the obtained governing kth -order problem for analytical solution $u_k(x,t)$ is defined by

$$L(u_k(x,t))=L(u_{k-1}(x,t))+C_kN_0(u_0(x,t))+\sum_{j=1}^{k-1}C_j\left[L(u_{k-j}(x,t))+N_{k-j}\left(u_0(x,t),u_1(x,t),...,u_{k-j}(x,t)\right)\right], \qquad (6)$$

$$k=2,3,...$$

$$B\left(u_k(x,t),\frac{\partial u_k(x,t)}{\partial t}\right)=0,$$

where $N_{k-j}\left(u_0(x,t),u_1(x,t),.........,u_{k-j}(x,t)\right)$ is the coefficient of p^{k-j} in the expansion of $N\left(F(x,t;p)\right)$ with respect the embedding parameter p and

$$N\left(F(x,t;p,C_1,C_2,...)\right)=N_0(u_0(x,t))+\sum_{k=1}^{\infty}N_k(u_0,u_1,......,u_k)p^k.$$

It may be noted that the solution $u_k(x,t),\ k\geq 0$ are governed by the linear equations, which is the simplest part of the Eq.(3) and with the linear boundary conditions that come from original problem (3), which can be easily solved. Now the interesting result of the above defined homotopy is that the series (6) is convergent at $p=1$ so that

$$\tilde{u}(x,t;C_1,C_2,...) = u_0(x,t) + \sum_{k\geq 1} u_k(x,t;C_1,C_2,...,C_k).$$

Generally speaking, the solution of the Eq.(3) can be determined approximately in the form:

$$\tilde{u}(x,t;C_1,C_2,...) = u_0(x,t) + \sum_{k=1}^{m} u_k(x,t;C_1,C_2,...,C_k). \quad (8)$$

The residual of Eq. (3) is obtained by substituting Eq. (8) into Eq. (3), we have

$$R(x,t;C_1,C_2,..,) = T(\tilde{u}(x,t;C_1,C_2,...)) + f(x,t). \quad (9)$$

If residual $R(x,t;\, C_1,C_2...) = 0$, then $\tilde{u}(x,t;C_1,C_2,...)$ will be the exact solution of Eq. (3).

Generally, it does not happen, particularly in nonlinear differential equations. To obtain the optimal values of auxiliary constants $C_1,C_2,C_3,...$, there are many methods like Galerkin's Method, Ritz Method, Least Squares Method and Collocation Method for to find the values of $C_1,C_2,C_3,....$

The Least square method in the above announced methods, one can apply as follows:

$$J(C_1,C_2,C_3,...) = \int_0^t \int_\Omega R^2(x,t;C_1,C_2,...)dxdt,$$

and the values of constants $C_1,C_2,C_3,...$ can be optimally identified from the following conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \frac{\partial J}{\partial C_3} = ... = 0.$$

For the values of $C_1,C_2,...$, the approximate solution $\tilde{u}(x,t;C_1,C_2,...)$ of Eq.(3) is well determined. The values of $C_1,C_2,C_3,...,C_m$ can be determined in another way (mentioned in [17] at Eq.(15)) as follows: For example, if $h_i \in \Omega,\ i=1,2,3,...,m$ and substituting h_i into Eq.(9), we obtain the equations

$$R(h_1,C_1,C_2,...,C_m) = R(h_2,C_1,C_2,...,C_m) = ... = R(h_m,C_1,C_2,...,C_m) = 0,$$

at any time *t*.

It can be observed by the application of OHAM that the general auxiliary function $H(p)$ is useful for convergence, which depends upon the values of $C_1,C_2,C_3,...,C_k,...$ can be optimally find by one of the above announced methods and is very useful to minimize the error.

RESULTS

In this section we present the results of algorithm outlined in the previous section. The accuracy of the OHAM is measured in terms of maximum error norm L_∞ defined as:

$$L_\infty = \max_j \left|(u)_j - (\tilde{u})_j\right|,$$

where u and $\tilde{u}$ represent the exact and approximate solutions respectively.

EXAMPLE-1

We consider the Generalized Burgers'-Huxley equation [23] given in (1). The exact solitary wave solution of Eq. (1) is given by [28], (9)

$$u(x,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\omega_1(x-\omega_2 t))\right)^{\frac{1}{\delta}}, x\in\Omega, t>0, \quad (10)$$

where

$$\omega_1 = \frac{-\alpha\,\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma \text{ and}$$

$$\omega_2 = \frac{\alpha\ \gamma}{(1+\delta)} - \frac{(1+\delta-\gamma)\left(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)}\right)}{2(1+\delta)},$$

α,β,δ and γ are constants so that $\beta\geq 0,\ \delta>0$, $\gamma\in(0,1)$.

For $\Omega=[0,1]$ and $f(x,t)=0$, the initial and boundary conditions of the Eq. (1) are given by

$$u(x,0) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1 x)\right)^{1/\delta},$$

$$u(0,t) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1\omega_2 t)\right)^{1/\delta},$$

$$u(1,t) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1(1-\omega_2 t))\right)^{1/\delta}.$$

Once again by OHAM formulation, one can choose L and N for Eq. (1) such as

$$L(F(x,t;p)) = \frac{\partial^2}{\partial x^2}F(x,t;p),$$

$$N(F(x,t;p)) = \beta F(x,t;p)\left(1-F^\delta(x,t;p)\right)\left(F^\delta(x,t;p)-\gamma\right) - \frac{\partial}{\partial t}F(x,t;p) - \alpha F^\delta(x,t;p)\frac{\partial}{\partial x}F(x,t;p),$$

with initial and boundary conditions are (10)

$$F(x,0;p) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1 x)\right)^{1/\delta},$$

$$F(0,t;p) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1\omega_2 t)\right)^{1/\delta},$$

$$F(1,t;p) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1(1-\omega_2 t))\right)^{1/\delta}.$$

With the help of selected L and N of the Eq. (1), we can generate series of problems and first problem of this series is zeroth-order problem is defined by

$$\frac{\partial^2}{\partial x^2}u_0(x,t) = 0, \quad (11)$$

$$u_0(0,t) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1\omega_2 t)\right)^{1/\delta}, \qquad (12)$$
$$u_0(1,t) = \left(\tfrac{\gamma}{2} + \tfrac{\gamma}{2}\tanh(\omega_1(1-\omega_2 t))\right)^{1/\delta}.$$

The solution of Eqs. (11)-(12) is given by

$$u_0(x,t) = \left(\tfrac{\gamma}{2} - \tfrac{\gamma}{2}\tanh(\omega_1\omega_2 t)\right)^{1/\delta} +$$
$$x\left(\left(\tfrac{\gamma}{2} - \tfrac{\gamma}{2}\tanh\left(\omega_1\left(\omega_2 t - 1\right)\right)\right)^{1/\delta} - \left(\tfrac{\gamma}{2} - \tfrac{\gamma}{2}\tanh(\omega_1\omega_2 t)\right)^{1/\delta}\right),$$

The first order problem is defined as

$$\frac{\partial^2}{\partial x^2}u_1(x,t;C_1) = C_1 N_0\left(u_0(x,t)\right) =$$
$$C_1[\beta\, u_0(x,t)\left(1-u_0^{\delta}(x,t)\right)\left(u_0^{\delta}(x,t)-\gamma\right) -$$
$$\frac{\partial}{\partial t}u_0(x,t) - \alpha\ u_0^{\delta}(x,t)\frac{\partial}{\partial x}u_0(x,t)],$$
$$u_1(0,t) = 0,\ u_1(1,t) = 0.$$

Solving this first order problem, we obtain $u_1(x,\ t;C_1)$.

The second order problem is defined as

$$\frac{\partial^2}{\partial x^2}u_2(x,t;C_1,C_2) = C_2 N_0(u_0(x,t)) + C_1 N_1(u_0,u_1) +$$
$$(1+C_1)L\left(u_1(x,t;C_1)\right) =$$
$$C_2[\beta\, u_0(x,t)\left(1-u_0^{\delta}(x,t)\right)\left(u_0^{\delta}(x,t)-\gamma\right) - \frac{\partial}{\partial t}u_0(x,t)$$
$$-\alpha\ u_0^{\delta}(x,t)\frac{\partial}{\partial x}u_0(x,t)] + C_1[\beta\, u_1(x,t;C_1)\left(1-u_1^{\delta}(x,t;C_1)\right)$$
$$\left(u_1^{\delta}(x,t;C_1)-\gamma\right) - \frac{\partial}{\partial t}u_1(x,t;C_1) - \alpha\ u_1^{\delta}(x,t;C_1)\frac{\partial}{\partial x}u_1(x,t;C_1)$$
$$+(1+C_1)\frac{\partial}{\partial x^2}u_1(x,t;C_1),\ u_2(0,t)=0,\ \ u_2(1,t)=0.$$

The solution of the second order problem gives $u_2(x,\ t;C_1,C_2)$.

The second order approximate solution of Eq. (1) is follow:

$$\tilde{u}(x,t;C_1,C_2) = u_0(x,t) + u_1(x,t;C_1) + u_2(x,t;C_1,C_2). \quad (13)$$

For the Residual of Eq.(1), we have

$$R(x,t;C_1,C_2) = \frac{\partial^2}{\partial x^2}\tilde{u}(x,t;C_1,C_2) +$$
$$\beta\ \tilde{u}(x,t;C_1,C_2)\left(1-\tilde{u}^{\delta}(x,t;C_1,C_2)\right)\left(\tilde{u}^{\delta}(x,t;C_1,C_2)-\gamma\right) -$$
$$\frac{\partial}{\partial t}\tilde{u}(x,t;C_1,C_2) - \alpha\ \tilde{u}^{\delta}(x,t;C_1,C_2)\frac{\partial}{\partial x}\tilde{u}(x,t;C_1,C_2).$$

To find the constants C_1 and C_2, we use the procedure given in the previous section, which gives the following values: $C_1 = -0.974338377203654$, $C_2 = -0.05859856342204$.

By using these values of auxiliary constants into Eq. (13), we get the second order approximate solution. The absolute errors of the second order approximate solution for various values of the parameter $\alpha, \beta, \gamma, \delta$ are reported in Tables 1 and 2. In Table 1, we have compared the results obtained by OHAM with those given in [11]. It can be noted from Table 1 that the results obtained by the present method are more accurate than those given in [11]. In Fig.1, we have shown exact and OHAM solutions corresponding to $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$.

Table 1: The absolute error of approximate solution by OHAM with the exact solution of Example-1 for $\alpha = \beta = \delta = 1,\ \gamma = 0.001$.

T	X	$\tilde{u}\times 10^3$	$u\times 10^3$	$L_\infty \times 10^8$	$L_\infty \times 10^7$
				OHAM	ADM [11]
0.05	0.1	0.500003	0.500019	1.60	1.93
	0.5	0.500021	0.500069	4.73	1.93
	0.9	0.500101	0.500119	1.81	1.93
0.1	0.1	0.500009	0.500025	1.60	3.87
	0.5	0.500028	0.500075	4.73	3.87
	0.9	0.500107	0.500125	1.81	3.87
1	0.1	0.500121	0.500137	1.60	38.8
	0.5	0.500140	0.500187	4.73	38.8
	0.9	0.500219	0.500237	1.81	38.8
5	0.1	0.500621	0.500637	1.60	-
	0.5	0.500640	0.500687	4.73	-
	0.9	0.500719	0.500737	1.81	-
50	0.1	0.506240	0.506256	1.60	-
	0.5	0.506259	0.506306	4.73	-
	0.9	0.506338	0.506356	1.81.	-

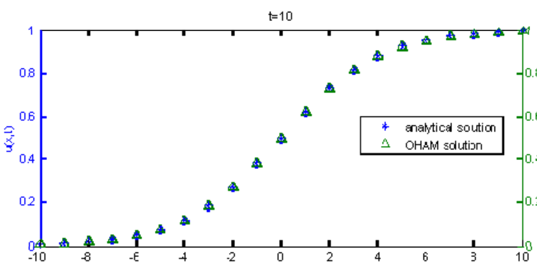

Fig. 1: Approximate and exact solution for Example-1

Table 2: The absolute error of approximate solution by OHAM with the exact solution of Example-1 for $\alpha = 0.001,\ \beta = 0.001,\ \delta = 1,\ \gamma = 0.001$.

T	X	$\tilde{u}\times 10^3$	$u\times 10^3$	$L_\infty \times 10^8$
0.005	0.1	0.500000	0.500000	2.11
	0.5	0.500003	0.500003	6.24
	0.9	0.500005	0.500005	2.39
0.01	0.1	0.500000	0.500000	2.11
	0.5	0.500003	0.500003	6.24
	0.9	0.500005	0.500005	2.39
5	0.1	0.500002	0.500002	2.11
	0.5	0.500004	0.500004	6.24
	0.9	0.500006	0.500006	2.39
50	0.1	0.500013	0.500013	2.11
	0.5	0.500015	0.500015	6.24
	0.9	0.500017	0.500017	2.39

Table 3: The absolute error of approximate solution by OHAM with the exact solution of Example-2 for $\alpha = 0,\ \beta = \delta = 1,\ \gamma = 0.001$.

T	X	$\tilde{u} \times 10^3$	$u \times 10^3$	$L_\infty \times 10^8$	$L_\infty \times 10^7$
				OHAM	ADM [11]
0.05	0.1	0.500009	0.500030	2.13	1.88
	0.5	0.500038	0.500101	6.31	1.87
	0.9	0.500147	0.500172	2.41	1.87
0.1	0.1	0.500021	0.500043	2.13	3.75
	0.5	0.500050	0.500113	6.31	3.75
	0.9	0.500160	0.500184	2.41	3.75
1	0.1	0.500246	0.500268	2.13	37.5
	0.5	0.500275	0.500338	6.31	37.5
	0.9	0.500385	0.500409	2.41	37.5
5	0.1	0.501246	0.501267	2.13	-
	0.5	0.501275	0.501338	6.31	-
	0.9	0.501384	0.501408	2.41	-
50	0.1	0.512488	0.512509	2.13	-
	0.5	0.512516	0.512579	6.30	-
	0.9	0.512626	0.512650	2.41	-

EXAMPLE-2

When $\alpha = 0, \delta = 1$, Eq. (1) reduces to the Huxley equation [11-12, 23, 25]. The OHAM results at the selected points of the domain are shown in Table 3 for $\beta = 1$ and $\gamma = 0.001$.

CONCLUSION

In this paper, an optimal homotopy asymptotic method is used for the approximate solution of Generalized Burger's-Huxley equation and its variants Burgers-Huxley equation and Huxley equation. Excellent accuracy is obtained in comparison with exact solution while better accuracy than adomian decomposition method is obtained. The results exhibit that the present method can be applied for the solution of this class of partial differential equations.

In future work we intend to develop an integration of our approaches with formal specification language to develop a linkage with computer modeling. Formal methods are languages based on discrete mathematics used for many applications [31-43].

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