

Solutions of Twelfth -Order Boundary Value Problems Using Polynomial Spline in Off Step Points

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Abstract: We use Polynomial spline functions in off step points to develop a numerical method for the solutions of twelfth order boundary value problems. We show that the present method gives an approximation which are better than those produced by other finite difference and spline methods. Two numerical examples are given to illustrate practical usefulness of our method.

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1 Introduction

We consider twelfth -order boundary-value problem of type

$$y^{(12)}(x) + f(x)y(x) = g(x), \quad x \in [a, b] \quad (1)$$

$$y(a) = \alpha_0, y^{(1)}(a) = \alpha_1, y^{(2)}(a) = \alpha_2, y^{(3)}(a) = \alpha_3, y^{(4)}(a) = \alpha_4, y^{(5)}(a) = \alpha_5, \\ y(b) = \beta_0, y^{(1)}(b) = \beta_1, y^{(2)}(b) = \beta_2, y^{(3)}(b) = \beta_3, y^{(4)}(b) = \beta_4, y^{(5)}(b) = \beta_5 \quad (2)$$

where α_i, β_i for $i = 0, 1, 2, 3, 4, 5$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$.

Twizell et al.[1] developed numerical methods for 8th-,10th-,and 12th-order eigenvalue problems arising in thermal instability. Siddiqi and Twizell[2] presented the solutions of 12th order boundary value problems using the 12th degree spline, respectively. Siddiqi and Akram[3] developed the solution of 12th-order boundary value problems using non-polynomial spline. Siddiqi and Akram[4] presented the solution of 12th-order boundary value problem by using thirteen degree spline.

In this paper we used polynomial spline approximation in off step points to develop a family of new numerical methods to smooth approximations to the solution of 12th-order differential equation. The method developed is observed to be better than that developed by Siddiqi et al [3], as discussed in Examples 1 and 2. In this paper, in Section 2, the new polynomial spline methods are developed for solving equation (1) along with boundary condition(2). The boundary formulas are develop in Section 3. In Section 4, the polynomial spline solution of the BVP (1),(2) is determined and in Section 5 numerical experiment,

discussion and comparison with other known methods, are give.

2 Numerical methods

Let $S_i(x)$ be the polynomial spline defined on $[a, b]$ as:

$$S_i(x) = a(x-x_i)^{13} + b(x-x_i)^{12} + c_i(x-x_i)^{11} + d_i(x-x_i)^{10} + e_i(x-x_i)^9 + f_i^*(x-x_i)^8 + \\ g_i^*(x-x_i)^7 + o_i(x-x_i)^6 + p_i(x-x_i)^5 + q_i(x-x_i)^4 + u_i(x-x_i)^3 + z_i^*(x-x_i)^2 + r_i(x-x_i) + u_i \quad (3)$$

$$x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], i = 0, 1, 2, \dots, n-1 \text{ and } x_0 = a, x_n = b,$$

$$\text{Where } h = \frac{b-a}{n} \text{ and } x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h, i = 1, 2, 3, \dots, n$$

The spline S is defined in terms of its 1td, 2th, 3th, 4th, 5th and 12th derivatives and we denote these values at knots as:

$$S_i(x_{i-\frac{1}{2}}) = y_{i-\frac{1}{2}}, S_i'(x_{i-\frac{1}{2}}) = m_{i-\frac{1}{2}}, S_i''(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}}, S_i'''(x_{i-\frac{1}{2}}) = z_{i-\frac{1}{2}}, S_i^{(4)}(x_{i-\frac{1}{2}}) = V_{i-\frac{1}{2}}$$

$$, S_i^{(5)}(x_{i-\frac{1}{2}}) = w_{i-\frac{1}{2}}, S_i^{(12)}(x_{i-\frac{1}{2}}) = L_{i-\frac{1}{2}},$$

$$S_i(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}}, S_i'(x_{i+\frac{1}{2}}) = m_{i+\frac{1}{2}}, S_i''(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}}, S_i'''(x_{i+\frac{1}{2}}) = z_{i+\frac{1}{2}}, S_i^{(4)}(x_{i+\frac{1}{2}}) = V_{i+\frac{1}{2}}$$

$$S_i^{(5)}(x_{i+\frac{1}{2}}) = w_{i+\frac{1}{2}}, S_i^{(12)}(x_{i+\frac{1}{2}}) = L_{i+\frac{1}{2}}$$

$$\text{For } i = 1, 2, \dots, n. \quad (4)$$

Assuming $y(x)$ to be the exact solution of the boundary value problem (1)

and y_i be an approximation to $y(x_i)$, obtained by the spline $S_i(x)$, we can obtained the coefficients in (3) in the following form

$$\begin{aligned}
a_i &= \frac{-1}{6227020800} h (L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}), \quad b_i = \frac{1}{958003200} (L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}), \\
c_i &= \frac{1}{4151347200} [h^{12}(L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}) + 34594560(15120h(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + 3360h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + \\
&\quad 30h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} + w_{i+\frac{1}{2}}) + 30240(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}}) + 420h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}))] , \\
d_i &= \frac{1}{638668800h^9} [h^{11}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 2661120(1680(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}) + h(840(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + \\
&\quad h(20h(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + h^2(w_{i-\frac{1}{2}} - w_{i+\frac{1}{2}}) + 180(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})))] , \\
e_i &= \frac{1}{6642155520h^9} [-h^{12}(L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}) - 69189120(18480h(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + 4080h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + \\
&\quad 30h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} + w_{i+\frac{1}{2}}) + 36960(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}}) + 500h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}))] , \\
f_i &= \frac{1}{1021870080h^7} [h^{11}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 5322240(2160(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}) + h(1080(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + \\
&\quad h(24h(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + h^2(w_{i-\frac{1}{2}} - w_{i+\frac{1}{2}}) + 228(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})))] , \\
g_i &= \frac{1}{1992646656h^7} [h^{12}(L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}) + 103783680(23760h(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + 5184h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + \\
&\quad 38h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} + w_{i+\frac{1}{2}}) + 47520(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}}) + 612h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}))] , \\
o_i &= \frac{-1}{3065610240h^5} [h^{11}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 7983360(3024(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}) + h(1512(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + \\
&\quad h(28h(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + h^2(w_{i-\frac{1}{2}} - w_{i+\frac{1}{2}}) + 308(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})))] , \\
p_i &= \frac{1}{1062744880h^5} [-h^{12}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 138378240(33264h(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + 7056h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + \\
&\quad 42h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} + w_{i+\frac{1}{2}}) + 66528(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}}) + 756h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}))] , \\
q_i &= \frac{1}{16349921280h^3} [h^{11}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 10644480(5040(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}) + h(2520(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + \\
&\quad h(32h(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + h^2(w_{i-\frac{1}{2}} - w_{i+\frac{1}{2}}) + 420(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})))] , \\
u_i &= \frac{1}{1062744880h^3} [h^{12}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 172972800(55440h(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + 10080h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + \\
&\quad 46h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} + w_{i+\frac{1}{2}}) + 110880(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}}) + 932h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}))] , \\
z_i &= \frac{-1}{163499212800h} [h^{11}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 13305600(15120(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}) + h(4488(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + \\
&\quad h(36h(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + h^2(w_{i-\frac{1}{2}} - w_{i+\frac{1}{2}}) + 546(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})))] , \\
r_i &= \frac{1}{2550587710800h} [-h^{12}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 207567360(104880h(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + 14640h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + \\
&\quad 50h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} + w_{i+\frac{1}{2}}) + 332640(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}}) + 1140h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}))] , \\
t_i &= \frac{1}{3923981107200h} [h^{12}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + 15966720(46320(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}) + 7800h^2(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + \\
&\quad 40h^4(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + h^5(w_{i-\frac{1}{2}} - w_{i+\frac{1}{2}}) + 122880(y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}}) + 740h^3(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}}))] ,
\end{aligned}$$

Assuming $y(x)$ to be the exact solution of the boundary value problem (1) and y_i be an approximation to $y(x_i)$ using the continuity conditions ($S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ where $\mu = 6, 7, 8, 9, 10$ and 11), we obtain the following spline relations:

$$\begin{aligned} & (y_{i-\frac{13}{2}} + y_{i+\frac{11}{2}}) - 12(y_{i-\frac{11}{2}} + y_{i+\frac{9}{2}}) + 66(y_{i-\frac{9}{2}} + y_{i+\frac{7}{2}}) - 220(y_{i-\frac{7}{2}} + y_{i+\frac{5}{2}}) + 495(y_{i-\frac{5}{2}} + y_{i+\frac{3}{2}}) - \\ & 792(y_{i-\frac{3}{2}} + y_{i+\frac{1}{2}}) + 924y_{i-\frac{1}{2}} = \frac{h^{12}}{6227020800} [(L_{i-\frac{13}{2}} + L_{i+\frac{11}{2}}) + 8178(L_{i-\frac{11}{2}} + L_{i+\frac{9}{2}}) + \\ & 1479726(L_{i-\frac{9}{2}} + L_{i+\frac{7}{2}}) + 4553345(L_{i-\frac{7}{2}} + L_{i+\frac{5}{2}}) + 423281535(L_{i-\frac{5}{2}} + L_{i+\frac{3}{2}}) + \\ & 1505621508(L_{i-\frac{3}{2}} + L_{i+\frac{1}{2}}) + 2275172004L_{i-\frac{1}{2}}], \quad i = 7, 8, \dots, n-7. \quad (5) \end{aligned}$$

3 Development of the boundary formulas

Liner system equation (5) consist of $(n-1)$ unknown, so that to obtain unique solution we need twelfth more equations to be associate with equation (5) so that we can develop the boundary formulas of different orders, but for sake of briefness here we develop the twelfth order boundary formulas so that we define the following identity:

$$w'_0 y_0 + \sum_{i=0}^7 a'_i y_{i+\frac{1}{2}} + c' h y'_0 + d' h^2 y''_0 + e' h^3 y'''_0 + u' h^4 y^{(4)}_0 + p' h^5 y^{(5)}_0 = h^{12} \sum_{i=0}^9 b'_i y^{(12)}_{i+\frac{1}{2}} \quad (6)$$

$$w''_0 y_0 + \sum_{i=0}^8 a''_i y_{i+\frac{1}{2}} + c'' h y'_0 + d'' h^2 y''_0 + e'' h^3 y'''_0 + u'' h^4 y^{(4)}_0 + p'' h^5 y^{(5)}_0 = h^{12} \sum_{i=0}^{10} b''_i y^{(12)}_{i+\frac{1}{2}} \quad (7)$$

$$w'''_0 y_0 + \sum_{i=0}^9 a'''_i y_{i+\frac{1}{2}} + c''' h y'_0 + d''' h^2 y''_0 + e''' h^3 y'''_0 + u''' h^4 y^{(4)}_0 + p''' h^5 y^{(5)}_0 = h^{12} \sum_{i=0}^{11} b'''_i y^{(12)}_{i+\frac{1}{2}} \quad (8)$$

$$w^{(4)}_0 y_0 + \sum_{i=0}^{10} a^{(4)}_i y_{i+\frac{1}{2}} + c^{(4)} h y'_0 + d^{(4)} h^2 y''_0 + e^{(4)} h^3 y'''_0 + u^{(4)} h^4 y^{(4)}_0 + p^{(4)} h^5 y^{(5)}_0 = h^{12} \sum_{i=0}^{12} b^{(4)}_i y^{(12)}_{i+\frac{1}{2}} \quad (9)$$

$$w^{(5)}_0 y_0 + \sum_{i=0}^{11} a^{(5)}_i y_{i+\frac{1}{2}} + c^{(5)} h y'_0 + d^{(5)} h^2 y''_0 + e^{(5)} h^3 y'''_0 + u^{(5)} h^4 y^{(4)}_0 + p^{(5)} h^5 y^{(5)}_0 = h^{12} \sum_{i=0}^{13} b^{(5)}_i y^{(12)}_{i+\frac{1}{2}} \quad (10)$$

$$\dot{w}_0 y_0 + \sum_{i=0}^{12} \dot{a}_i y_{i+\frac{1}{2}} + \dot{c} h y'_0 + \dot{d} h^2 y''_0 + \dot{e} h^3 y'''_0 + \dot{u} h^4 y^{(4)}_0 + \dot{p} h^5 y^{(5)}_0 = h^{12} \sum_{i=0}^{14} \dot{b}_i y^{(12)}_{i+\frac{1}{2}} \quad (11)$$

$$\ddot{w}_n y_n + \sum_{i=0}^{12} \ddot{a}_i y_{i+n-\frac{25}{2}} + \ddot{c} h y'_n + \ddot{d} h^2 y''_n + \ddot{e} h^3 y'''_n + \ddot{u} h^4 y^{(4)}_n + \ddot{p} h^5 y^{(5)}_n = h^{12} \sum_{i=0}^{14} \ddot{b}_i y^{(10)}_{i+n-\frac{29}{2}} \quad (12)$$

$$\ddot{\ddot{w}}_n y_n + \sum_{i=0}^{11} \ddot{\ddot{a}}_i y_{i+n-\frac{23}{2}} + \ddot{\ddot{c}} h y'_n + \ddot{\ddot{d}} h^2 y''_n + \ddot{\ddot{e}} h^3 y'''_n + \ddot{\ddot{u}} h^4 y^{(4)}_n + \ddot{\ddot{p}} h^5 y^{(5)}_n = h^{12} \sum_{j=0}^{13} \ddot{\ddot{b}}_j y^{(12)}_{i+n-\frac{27}{2}} \quad (13)$$

$$w^*_0 y_n + \sum_{i=0}^{10} a^*_i y_{i+n-\frac{21}{2}} + c^* h y'_n + d^* h^2 y''_n + e^* h^3 y'''_n + u^* h^4 y^{(4)}_n + p^* h^5 y^{(5)}_n = h^{12} \sum_{j=0}^{12} b^*_i y^{(12)}_{i+n-\frac{25}{2}} \quad (14)$$

$$\bar{w}_0 y_n + \sum_{i=0}^9 \bar{a}_i y_{i+n-\frac{19}{2}} + \bar{c} h y'_n + \bar{d} h^2 y''_n + \bar{e} h^3 y'''_n + \bar{u} h^4 y^{(4)}_n + \bar{p} h^5 y^{(5)}_n = h^{12} \sum_{i=0}^{11} \bar{b}_i y^{(12)}_{i+n-\frac{23}{2}} \quad (15)$$

$$\tilde{w}_0 y_n + \sum_{i=0}^8 \tilde{a}_i y_{i+n-\frac{17}{2}} + \tilde{c} h y'_n + \tilde{d} h^2 y''_n + \tilde{e} h^3 y'''_n + \tilde{u} h^4 y^{(4)}_n + \tilde{p} h^5 y^{(5)}_n = h^{12} \sum_{i=0}^{10} \tilde{b}_i y^{(12)}_{i+n-\frac{21}{2}} \quad (16)$$

$$\tilde{\tilde{w}}_0 y_n + \sum_{i=0}^7 \tilde{\tilde{a}}_i y_{i+n-\frac{15}{2}} + \tilde{\tilde{c}} h y'_n + \tilde{\tilde{d}} h^2 y''_n + \tilde{\tilde{e}} h^3 y'''_n + \tilde{\tilde{u}} h^4 y^{(4)}_n + \tilde{\tilde{p}} h^5 y^{(5)}_n = h^{12} \sum_{i=0}^9 \tilde{\tilde{b}}_i y^{(12)}_{i+n-\frac{19}{2}} \quad (17)$$

$$\begin{aligned}
c_{\frac{1}{2}} &= -w_0' y_0 - c' h y_0' - d' h^2 y_0'' - e' h^3 y_0''' - u' h^4 y_0^{(4)} - p' h^5 y_0^{(5)} + h^{12} \sum_{i=0}^9 b_i' g_{i+\frac{1}{2}}, \\
c_{\frac{3}{2}} &= -w_0'' y_0 - c'' h y_0' - d'' h^2 y_0'' - e'' h^3 y_0''' - u'' h^4 y_0^{(4)} - p'' h^5 y_0^{(5)} + h^{12} \sum_{i=0}^{10} b_i'' g_{i+\frac{1}{2}}, \\
c_{\frac{5}{2}} &= -w_0''' y_0 - c''' h y_0' - d''' h^2 y_0'' - e''' h^3 y_0''' - u''' h^4 y_0^{(4)} - p''' h^5 y_0^{(5)} + h^{12} \sum_{i=0}^{11} b_i''' g_{i+\frac{1}{2}}, \\
c_{\frac{7}{2}} &= -w_0^{(4)} y_0 - c^{(4)} h y_0' - d^{(4)} h^2 y_0'' - e^{(4)} h^3 y_0''' - u^{(4)} h^4 y_0^{(4)} - p^{(4)} h^5 y_0^{(5)} + h^{12} \sum_{i=0}^{12} b_i^{(4)} g_{i+\frac{1}{2}}, \\
c_{\frac{9}{2}} &= -w_0^{(5)} y_0 - c^{(5)} h y_0' - d^{(5)} h^2 y_0'' - e^{(5)} h^3 y_0''' - u^{(5)} h^4 y_0^{(4)} - p^{(5)} h^5 y_0^{(5)} + h^{12} \sum_{i=0}^{13} b_i^{(5)} g_{i+\frac{1}{2}}, \\
c_{\frac{11}{2}} &= -\dot{w}_0 y_0 - \dot{c} h y_0' - \dot{d} h^2 y_0'' - \dot{e} h^3 y_0''' - \dot{u} h^4 y_0^{(4)} - \dot{p} h^5 y_0^{(5)} + h^{12} \sum_{i=0}^{14} \dot{b}_i g_{i+\frac{1}{2}}, \\
c_{i-\frac{1}{2}} &= h^{12} (\alpha g_{i-\frac{13}{2}} + \beta g_{i-\frac{11}{2}} + \gamma g_{i-\frac{9}{2}} + \delta g_{i-\frac{7}{2}} + \eta g_{i-\frac{5}{2}} + \mu g_{i-\frac{3}{2}} + \tau g_{i-\frac{1}{2}} + \mu g_{i+\frac{1}{2}} + \eta g_{i+\frac{3}{2}} + \\
&\quad \delta g_{i+\frac{5}{2}} + \gamma g_{i+\frac{7}{2}} + \beta g_{i+\frac{9}{2}} + \alpha g_{i+\frac{11}{2}}), \quad i = 7, 8, \dots, (n-7) \\
c_{n-\frac{11}{2}} &= -\ddot{w}_0 y_n - \ddot{c} h y_n' - \ddot{d} h^2 y_n'' - \ddot{e} h^3 y_n''' - \ddot{u} h^4 y_n^{(4)} - \ddot{p} h^5 y_n^{(5)} + h^{12} \sum_{i=0}^{14} \ddot{b}_i g_{i+n-\frac{29}{2}}, \\
c_{n-\frac{9}{2}} &= -\ddot{w}_0 y_n - \ddot{c} h y_n' - \ddot{d} h^2 y_n'' - \ddot{e} h^3 y_n''' - \ddot{u} h^4 y_n^{(4)} - \ddot{p} h^5 y_n^{(5)} + h^{12} \sum_{i=0}^{13} \ddot{b}_i g_{i+n-\frac{27}{2}}, \\
c_{n-\frac{7}{2}} &= -w_0^* y_n - c^* h y_n' - d^* h^2 y_n'' - e^* h^3 y_n''' - u^* h^4 y_n^{(4)} - p^* h^5 y_n^{(5)} + h^{12} \sum_{i=0}^{12} b_i^* g_{i+n-\frac{25}{2}}, \\
c_{n-\frac{5}{2}} &= -\bar{w}_0 y_n - \bar{c} h y_n' - \bar{d} h^2 y_n'' - \bar{e} h^3 y_n''' - \bar{u} h^4 y_n^{(4)} - \bar{p} h^5 y_n^{(4)} + h^{12} \sum_{i=0}^{11} \bar{b}_i g_{i+n-\frac{23}{2}}, \\
c_{n-\frac{3}{2}} &= -\tilde{w}_0 y_n - \tilde{c} h y_n' - \tilde{d} h^2 y_n'' - \tilde{e} h^3 y_n''' - \tilde{u} h^4 y_n^{(4)} - \tilde{p} h^5 y_n^{(5)} + h^{12} \sum_{i=0}^{10} \tilde{b}_i g_{i+n-\frac{21}{2}}, \\
c_{n-\frac{1}{2}} &= -\tilde{w}_0 y_n - \tilde{c} h y_n' - \tilde{d} h^2 y_n'' - \tilde{e} h^3 y_n''' - \tilde{u} h^4 y_n^{(4)} - \tilde{p} h^5 y_n^{(5)} + h^{12} \sum_{i=0}^9 \tilde{b}_i g_{i+n-\frac{19}{2}},
\end{aligned}$$

5 Numerical results

Example 1. We Consider the following boundary-value problem

$$\begin{aligned}
y^{(12)}(x) + xy(x) &= -(120 + 23x + x^3)e^x, \quad 0 \leq x \leq 1, \\
y(0) &= 0, \quad y(1) = 0, \\
y'(0) &= 1, \quad y'(1) = -e, \\
y''(0) &= 0, \quad y''(1) = -4e, \\
y'''(0) &= -3, \quad y'''(1) = -9e, \\
y^{(4)}(0) &= -8, \quad y^{(4)}(1) = -16e.
\end{aligned}$$

$$y^{(5)}(0) = -15, y^{(5)}(1) = -25e. \quad (15)$$

The analytic solution of the above system is $y(x) = x(1-x)e^x$. It is evident from Table 1 that the maximum errors in absolute values are less than those presented by [3].

Example 2. We Consider the following boundary-value problem

$$\begin{aligned}
y^{(12)}(x) - y(x) &= -12(2x \cos x + 1 \sin x), \quad -1 \leq x \leq 1, \\
y(-1) &= y(1) = 0, \\
y'(-1) &= y'(1) = 2 \sin(1),
\end{aligned}$$

$$\begin{aligned}
 y''(-1) &= -y''(1) = -4 \cos(1) - 2 \sin(1), \\
 y'''(-1) &= y'''(1) = 6 \cos(1) - 6 \sin(1), \\
 y^{(4)}(-1) &= -y^{(4)}(1) = 8 \cos(1) + 12 \sin(1), \\
 y^{(5)}(-1) &= y^{(5)}(1) = -20 \cos(1) + 10 \sin(1).
 \end{aligned}$$

The analytic solution of the above system is $y(x) = (x^2 - 1)\sin x$. It is evident from Table 2 that the maximum errors in absolute values are less than those presented by [3].

Conclusion

We approximate solution of the twelfth-order linear boundary-value problems by using polynomial spline. The new methods enable us to approximate the solution at every point of the range of integration. The method is compared with that developed by et al [3] considering the same examples. Tables 1-2 shows that our methods produced better result the sense that $\max|e_i| = \max|y(x_i) - y_i|$ is in comparison with the method in [3].

Table 1: Observed maximum absolute errors for example 1.

h	Our methods	[3]
1/9	3.84(-12)	7.38(-9)
1/18	8.98(-13)	-
1/27	2.04(-13)	-
1/36	7.58(-14)	-

Table 2: Observed maximum absolute errors for example 2.

h	Our methods	[3]
1/16	9.15(-10)	1.14(-7)
1/32	1.33(-10)	-
1/48	6.31(-11)	-
1/64	2.19(-11)	-

References

1. Twizell, E .Boutayeb,A .Djidjeli, K., "Numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability", Advances in Computational Mathematics, Vol.2,pp. 407-436, 1994.
2. Siddiqi, S.S, Twizell, E.H., "Spline solution of linear twelfth-order boundary value problems", Int. J.Comput.Appl.Math.Vol.78,PP.371-390, 1997.
3. Siddiqi, S.S, Akram,G., "Solution of 12th-order boundary value problems using non-polynomial spline technique ", Applied Mathematics and Computation, Vol.199,pp. 559-571, 2008.
4. Siddiqi, S.S, Akram,G., "Solution of tenth-order boundary value problems using eleventh degree spline", Applied Mathematics and Computation, Vol.182,pp. 1443-1453, 2006.
5. J. Rashidinia, R. Jalilian, K. Farajeyan. Spline approximate solution of eighth-order boundary-value problems. International Journal of Computer Mathematics, 2009, 86: 1319-1333.