# Higher-order asymptotic formula for the eigenvalues of Sturm-Liouville problem with indefinite weight function in the Neumann boundary condition

Farhad Dastmalchi Saei[1] , A. Jodayree Akbarfam[2]

1. Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran
2. Faculty of Mathematical Sciences Tabriz University, Tabriz, Iran
E-Mail: dastmalchi@iaut.ac.ir; E-Mail: akbarfam@ark.tabrizu.ac.ir

**Abstract:** In this paper ,we investigate the asymptotic behavior of the differential equation $y'' + (\lambda r(x) - q(x))y = 0,\ 0 \le x \le 1$. Where $[0,1]$ contains a finite number of zeros of $r(x)$, the so called turning points, $\lambda$ is a real parameter and the function $q(x)$ is bounded and integrable in $[0,1]$. Using a technique used previously in [7], we derive the higher-order asymptotic distribution of the positive eigenvalues associated with this equation for the Neumann problem(i.e, $y'(0) = y'(1) = 0$). In most differential equations with variable coefficient it is impossible to obtain an exact solution, so we want to obtain asymptotic distribution of the eigenvalues without solving equation.

[Farhad Dastmalchi Saei, A. Jodayree Akbarfam. **Higher-order asymptotic formula for the eigenvalues of Sturm-Liouville problem with indefinite weight function in the Neumann boundary condition.** *Life Sci J* 2012;9(4):2975-2979]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 437

**Keywords**: turning point, Sturm-Liouville, nondefinite, asymptotic distribution, eigenvalue.
AMS Sub. Classification: 34L20.

## 1. Introduction

We study the indefinite Sturm-Liouville spectral problem

$$y'' + (\lambda r(x) - q(x))y = 0,\quad a \le x \le b \quad (1)$$
$$y'(a) = y'(b) = 0,$$

defined on the interval $[a, b]$ where $\lambda$ is a real parameter, $r(x), q(x)$ are real and integrable on $[a, b]$; moreover,

$$\int_a^b \sqrt{r_+(t)}dt > 0, where\ r_+(x) = max\{r(x),0\}. \quad (2)$$

It follows from [2] that the spectrum of this problem is discrete and has no finite accumulation points; moreover, only finitely many eigenvalues lie the outside the real and imaginary axes. In what follows, we shall assume that $\lambda$ is a positive parameter. In [4] it was shown that the asymptotics of the eigenvalues is of the form

$$\lambda_n \sim \frac{n\pi}{\int_a^b \sqrt{r_+(t)}dt}. \quad (3)$$

Our goal is to refine the asymptotics under the additional assumptions of smoothness of the functions $r(x)$ and $q(x)$. In addition, we assume that $r(x)$ has a finite number of zeros, which are called turning points.

The outline of our paper is as follows. First, we find the asymptotics of eigenvalues for one turning point. Next, using a technique previously in [9], we derive the higher order asymptotic distribution of the positive eigenvalues in the case of two turning points. Finally, in the case of an arbitrary finite number of turning points it can be reduced to the two cases discussed above.

## 2. The case of one turning point

First, consider the case

$$r(x) = (x - x_v)^{l_v} h(x), h(x) > 0.$$

To simplify the formulas, we assume that $x$ varies on the closed interval with endpoints $a$ and $b$, where $r(a) < 0$ and $r(b) > 0$. The turning point $x_v$ lies between $a$ and $b$.

We distinguish four different types of turning points:

$$T_v = \begin{cases} I & if\, l_v is\ even\ and\ r(x) < 0 in [a,b] \\ II & if\, l_v is\ even\ and\ r(x) > 0 in [a,b] \\ III & if\, l_v is odd\ and\ r(x) < 0 in [x_v, b] \\ IV & if\, l_v is\ odd\ and\ r(x) > 0 in [x_v, b] \end{cases}$$

is called of type $x_v$. By Langer's transformation we can make zero of $r(x)$ the origin. To be specific, let us define the Langer's transformation $\xi(x)$ for different type of TP.

For a turning point of Type I :

$$\xi_I(x) = \begin{cases} -\{\int_x^{x_v} (-r)^{1/2}(t)dt\}^{\frac{2}{l+2}} & x \le x_v \\ -\{\int_{x_v}^{x} (-r(t))^{1/2}dt\}^{\frac{2}{l+2}} & x_v \le x. \end{cases} \quad (4)$$

For a turning point of Type II :

$$\xi_{II}(x)=\begin{cases}\{\int_x^{x_\nu} r^{1/2}(t)dt\}^{\frac{2}{\ell+2}} & x\le x_\nu\\ \{\int_{x_\nu}^{x} r(t)^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu\le x.\end{cases}\qquad(5)$$

For a turning point of Type III :

$$\xi_{III}(x)=\begin{cases}\{\int_x^{x_\nu} r^{1/2}(t)dt\}^{\frac{2}{\ell+2}} & x\le x_\nu\\ -\{\int_{x_\nu}^{x} (-r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu\le x.\end{cases}\qquad(6)$$

For a turning point of Type IV :

$$\xi_{IV}(x)=\begin{cases}-\{\int_x^{x_\nu} (-r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x< x_\nu\\ \{\int_{x_\nu}^{x} (r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu\le x.\end{cases}\qquad(7)$$

From [9] we rewrite showing the connection between the argument of complex valued solution of (1) in the interval containing one of the turning point say, $x_\nu$, and the argument of complex valued solution of Sturm-Liouville equation with one turning point in $x=0$ in the same interval. In fact the following result illustrates a crucial relationship between a general problem ((1)) with a turning point at $x_\nu$ to a transformed problem in which is mapped to $x=0$.We show that such a transformation preserves the argument of any fixed complex valued solution.

**Theorem 1**Let z be a strictly complex -valued solution of the differential equations

$$y''+(\rho^2 r(x)-q(x))y=0, x\in[0,1]\qquad(8)$$

and $W$ be a solution of

$$W''+(u^2(-1)^{M_\nu}\xi^{\ell_\nu}-R_\nu(\xi))W=0, \xi\in[c,d]\qquad(9)$$

then on the interval $[x_\nu-\varepsilon, x_\nu+\varepsilon]$

$arg W(\xi(x))=\arg z$,

where $r(x)=\prod_{j=1}^{n}(x-x_j)^{l_j}\phi_0(x)$ and

$$R_\nu(\xi)=\left(\frac{dx}{d\xi}\right)^{1/2}\frac{d^2}{d\xi^2}\left\{\frac{1}{\left(\frac{dx}{d\xi}\right)^{1/2}}\right\}+\left(\frac{dx}{d\xi}\right)^2 q(x(\xi)).$$

$M_k=$ the number of turning of type (III) or (IV) in $(x_k,1)$, or one can see that

$(-1)^{M_k}=(-1)^{l_n+\cdots+l_{k-1}}$, $c<0<d$,

$u^2=\frac{(\ell_\nu+2)^2}{4}\rho^2$,

Thetransformation $\xi(x)$ is Langer's transformation.

**Proof:** For proof see [9].

## 3.The main result

We begin by consolidating some results from [5,9] for completeness. For a complex-valued solution $\Omega(x,\lambda)$, of

$$y''+\lambda x^\alpha y=0,\qquad(E_0)$$

we form the logarithmic derivative $r_0(x,\lambda)=\Omega'(x,\lambda)/\Omega(x,\lambda)$, a quantity that exists for each $x\in[a,b]$ since the real and imaginary parts of $\Omega$ are linearly independent solution of $(E_0)$. The quantity $r_1(x,\lambda)$ is defined by setting

$$r_1(x,\lambda)=-\int_x^b q(t)e^{2\int_x^t r_0(s,\lambda)ds}dt,$$

while the $r_n(x,\lambda)$ are defined recursively $(for\ n\ge 1)$ by

$$r_{n+1}(x,\lambda)=\int_x^b r_n^2(x,\lambda)\exp(2\sum_{i=0}^{n}\int_x^t r_i(s,\lambda)ds)dt$$

It follows $(cf.[4])$ that the function

$$r(x,\lambda)=\sum_{n=0}^{\infty} r_n(x,\lambda):=S(x,\lambda)+iT(x,\lambda)$$

is a series solution (in $x$) of the Riccati equation

$$v'=q-\lambda x^\alpha-v^2$$

from which one can reconstruct solutions of (1) with Neumann condition

$y'(a)=y'(b)=0$ via the following result:

**Theorem 2**(see Harris-Talarico[4]) There exists $\lambda_0$ such that any real valued solution of

$$y''+(\lambda x^\alpha-q(x))y=0\qquad(10)$$

can be expressed as :

$$Z(x,\lambda)=c_1 e^{\int_a^x S(t,\lambda)dt}\cos(c_2+\int_a^x T(t,\lambda)dt)$$

for $x\in[a,b](a<0<b)$ and $|\lambda|\ge\lambda_0$ where $c_1,c_2\in\Re$.If $Z(.,\lambda)$ satisfies

$$y(a)\cos\gamma+y'(a)\sin\gamma=0\qquad(11)$$

then

$$c_2=:c_2^a=\begin{cases}\frac{\pi}{2} & if\ \gamma=0\\ \arctan\left(\frac{1}{T(a,\lambda)}S(a,\lambda)+\cot\gamma\right) & if\ \gamma\ne0\end{cases}\qquad(12)$$

Similarly,if $Z$ satisfies

$$y(b)\cos\beta+y'(b)\sin\beta=0\qquad(13)$$

then

$$c_2=:c_2^b=\begin{cases}n\pi+\frac{\pi}{2} & if\ \beta=0\\ n\pi+\arctan\left(\frac{1}{T(b,\lambda)}S(b,\lambda)+\cot\beta\right) & if\ \beta\ne0\end{cases}\qquad(14)$$

for all integer $n$.

It follows from (12) and (14) that the eigenvalues of (10),(11) and (13),i.e., our problem (1), are the values of $\lambda$ for which

$$c_2^a + \int_a^b T(t,\lambda)dt = c_2^b \qquad (15)$$

We see from $[3]$, that the asymptotic distribution of the eigenvalues of (10),(11) and (13) is therefore determined by the following transcendental equation:

$$\begin{aligned} n\pi + \arctan\left(\frac{1}{T(b,\lambda)}S(b,\lambda) + \cot\beta\right) &= \int_a^b T(t,\lambda)dt + \arctan\left(\frac{1}{T(a,\lambda)}S(a,\lambda) + \cot\gamma\right) \\ &= \Im\int_a^b r(t,\lambda)dt + \arctan\left(\frac{1}{T(b,\lambda)}S(b,\lambda) + \cot\gamma\right) \\ &= \Im\left(\int_a^b r_0(t,\lambda)dt + \int_a^b r_1(t,\lambda)dt + \cdots\right) \\ &+ \arctan\left(\frac{1}{T(a,\lambda)}S(a,\lambda) + \cot\gamma\right) \\ &= \arg\Omega(b,\lambda) - \arg\Omega(a,\lambda) - \frac{\pi}{2k}\int_0^b xq(x)J_v^2\left(k^{-1}\lambda^{\frac{1}{2}}x^k\right)dx + \cdots \\ &+ \arctan\left(\frac{1}{T(a,\lambda)}S(a,\lambda) + \cot\gamma\right). \end{aligned} \qquad (16)$$

Note that we use the following result from [8],

$$\Im\int_a^b r(t,\lambda)dt = \arg\Omega(b,\lambda) - \arg\Omega(a,\lambda) - \frac{\pi}{2k}\int_0^b xq(x)J_v^2(k^{-1}\lambda^{1/2}x^k)dx.$$

By applying the above relation to approximate eigenvalues in the case of $\gamma = \beta = \frac{\pi}{2}$.

**Theorem 3** Consider the differential equation $(1)$ on $[a,b]$ under condition $(2)$. Then the positive eigenvalues admit the following asymptotic representation:

$(a)$ Let $x_v$ be of type $IV$. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{3\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}\left(\frac{4(v-1)^2-1}{8\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(b)\right) + O\left(\frac{1}{n^2}\right)$$

(17)

where

$$H(b) = \int_{x_v}^b \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}\left(\tilde{r}^{-1/4}\right)\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

$(b)$ Let $x_v$ be of type $III$. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}\left(\frac{4(\gamma-1)^2-1}{8\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(a)\right) + O\left(\frac{1}{n^2}\right), \qquad (18)$$

where

$$H(a) = \int_a^{x_v} \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}\left(\tilde{r}^{-1/4}\right)\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

$(c)$ Let $x_v$ be of type $II$. Then

$$\sqrt{\lambda_n} = \frac{n\pi}{\int_a^b \sqrt{r(t)}dt} - \frac{1}{n\pi}\left\{\frac{4(v-1)^2-1}{8\int_a^{x_v} \sqrt{r(t)}dt} + \left\{\frac{4(v-1)^2-1}{8\int_{x_v}^b \sqrt{r(t)}dt} - \frac{1}{2}H(a) - \frac{1}{2}H(b)\right\} + O\left(\frac{1}{n^2}\right)$$

(19)

where $H(a)$ and $H(b)$ are defined above. Case1: $\gamma = 0, \beta = \frac{\pi}{2}$

$(a)$ Let $x_v$ be of type $IV$. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_-(t)}dt} - \frac{1}{n\pi}\left(\frac{4(v-1)^2-1}{3\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(b)\right) + O\left(\frac{1}{n^2}\right)$$

(20)

where

$$H(b) = \int_{x_v}^b \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}\left(\tilde{r}^{-1/4}\right)\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

$(b)$ Let $x_v$ be of type $III$. Then

$$\sqrt{\lambda_n} = \frac{n\pi + \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}\left(\frac{4(\gamma-1)^2-1}{8\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(a)\right) + O\left(\frac{1}{n^2}\right)$$

(21)

where

$$H(a) = \int_a^{x_v} \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}\left(\tilde{r}^{-1/4}\right)\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

$(c)$ Let $x_v$ be of type $II$. Then

$$\sqrt{\lambda_n} - \frac{n\pi}{\int_a^b \sqrt{r(t)}dt} - \frac{1}{n\pi}\left\{\frac{4(v-1)^2-1}{8\int_a^{x_v} \sqrt{r(t)}dt} + \left\{\frac{4(v-1)^2-1}{8\int_{x_v}^b \sqrt{r(t)}dt} - \frac{1}{2}H(a) - \frac{1}{2}H(b)\right\} + O\left(\frac{1}{n^2}\right),$$

(22)

where $H(a)$ and $H(b)$ are defined above.

**Proof:** For proof see [8,9].

**2. The cases of two and n turning points**

From now then, without losing generalization, we suppose that the coefficients $q(x)$ and $r(x)$ satisfy:

(i) $r(x)$ is real and has in [0, 1] n zeros $x_\nu$ of order $l_\nu \in$ N , $1 \le \nu \le n$ where $0 < x_1 < x_2 < \ldots < x_n < 1$.

(ii) The function $\Phi_0 : I \to R - \{0\}, x \to r(x)\prod_{\nu=1}^{n}(x - x_\nu)^{-l_\nu}$ is twice continuously differentiable.

(iii) $q(x)$ is bounded and integrable in I.

We shall use the symbol $\Omega_{IV}(\xi, u)$ to signify the complex-valued solution of

$$W'' + u^2 \xi^{l_\nu} W = 0,$$

where $\xi$ is corresponding Langer's transformation of turning point of type IV. We will use the symbols $\Omega_I(\xi, u)$, $\Omega_{II}(\xi, u)$ and $\Omega_{III}(\xi, u)$ in similar case.

Now we can derive the following results on the distribution of the eigenvalues of (1) with Neumann boundary condition .

We consider only the following case:

$r(0) < 0, r(1) < 0.$

**1.a** $T_1 = IV, T_2 = III.$

We suppose that the weight function $r(x)$ has in $[0,1]$ two zeros $x_1$ and $x_2$ where $x_1$ of type $IV$ and $x_3$ of type $III$ . By $(11)$ the distribution of positive eigenvalue satisfies :

$$n\pi\left(= \Im\int_0^1 \frac{y'}{y}dx\right) = \Im\left(\int_0^{\alpha_{12}}\frac{y'}{y}dx + \int_{\alpha_{12}}^1 \frac{y'}{y}dx\right) + \arctan\left(\frac{S(0,\lambda)}{T(0,\lambda)}\right) - \arctan\left(\frac{S(1,\lambda)}{T(1,\lambda)}\right)$$
$$= \rho\int_0^1 \sqrt{r_+(t)}\,dt + \frac{\pi}{2} - \frac{1}{4\rho}\left(\frac{8\nu_1 - 4}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{8\nu_2 - 4}{\int_{x_1}^{x_2}\sqrt{r(t)}dt}\right) + \frac{1}{4\rho}\left(\frac{4\nu_1^2 - 1}{\int_{x_1}^{x_2}\sqrt{r(t)}\,dt} + \frac{4\nu_2^2 - 1}{\int_{x_1}^{x_2}\sqrt{r(t)}\,dt}\right) - \frac{1}{2u}P(x_1, x_2) + O\left(\frac{1}{u^2}\right), \quad (23)$$

where $\alpha_{12} \in (x_1, x_2)$ is such that $\int_{x_1}^{\alpha_{12}}\sqrt{r(t)}dt = \int_{\alpha_{12}}^{x_2}\sqrt{r(t)}dt$ (the existence of $\alpha_{12}$ follows by Intermediate Value Theorem), and

$$P(x_1, x_2) = P_{IV}(x_1, \alpha_{12}) + P_{III}(x_2, \alpha_{12}) = \int_{x_1}^{\alpha_{12}} E_{IV}(x)dx + \int_{\alpha_{12}}^{x_2} E_{III}(x)dx,$$

$$P_{IV}(x_1, \alpha_{12}) = \int_0^{d_{12}} \frac{R_{IV}(\xi)}{\xi^{\frac{l}{2}}}d\xi = \int_{x_1}^{\alpha_{12}}\left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4})\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx := \int_{x_1}^{\alpha_{12}} E_{IV}(x)dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}, \xi(\alpha_{12}) = d_{12}, \xi(x_1) = 0, \xi(0) = c_{12}$$

$$P_{III}(x_2, \alpha_{12}) = \int_{\alpha_{12}}^{x_2}\left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4})\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx := \int_{\alpha_{12}}^{x_2} E_{III}(x)dx.$$

By inversion, we get

$$\rho_n = \frac{n\pi - \frac{\pi}{2}}{\int_0^1\sqrt{r_+(t)}dt} - \frac{1}{n\pi}\left\{\frac{[4(\nu_1-1)^2-1]+[4(\nu_2-1)^2-1]}{4\int_{x_1}^{x_2}\sqrt{r(t)}dt} - \frac{1}{2}P(x_1,x_2)\right\} + O\left(\frac{1}{n^2}\right) \quad (24)$$

**2.a** $T_1 = IV$ , $T_2 = T_3 = \ldots = T_{n-1} = II$ , $T_n = III.$

By applying the same method and using theorem $(1)$, $(2)$ we get :

$$n\pi = \rho\int_0^1\sqrt{r_+(t)}dt - \frac{(n-1)\pi}{2} + \frac{n\pi}{2} + \frac{1}{4\rho}\left(\frac{4(\nu_1-1)^2-1}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{4(\nu_2-1)^2-1}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{4(\nu_2-1)^2-1}{\int_{x_2}^{x_3}\sqrt{r(t)}dt} + \frac{4(\nu_3-1)^2-1}{\int_{x_2}^{x_3}\sqrt{r(t)}dt} + \frac{4(\nu_3-1)^2-1}{\int_{x_3}^{x_4}\sqrt{r(t)}dt} + \ldots + \frac{4(\nu_{n-1}-1)^2-1}{\int_{x_{n-2}}^{x_{n-1}}\sqrt{r(t)}dt} + \frac{4(\nu_{n-1}-1)^2-1}{\int_{x_{n-1}}^{x_n}\sqrt{r(t)}dt} + \frac{4\nu_n^2-1}{\int_{x_{n-1}}^{x_n}\sqrt{r(t)}dt}\right) - \frac{1}{2u}P(x_1, x_2, \ldots, x_n) + O\left(\frac{1}{u^2}\right).$$

Where

$$P(x_1,x_2,\dots,x_n) = \int_{x_1}^{\alpha_{12}} E_{IV}(x)dx + \sum_{i=1}^{n-2}\int_{\alpha_{i(i+1)}}^{x_{i+1}} E_{II}^{-}(x) + \sum_{i=1}^{n-2}\int_{x_{i+1}}^{\alpha_{i+1(i+2)}} E_{II}^{+}(x) + \int_{\alpha_{(n-1)n}}^{1} E_{III}(x)dx$$

and by inversion :

$$\rho_n = \frac{n\pi - \frac{\pi}{2}}{\int_0^1 \sqrt{r_+(x)}dx} - \frac{1}{n\pi}\Big[\frac{[4(v_1-1)^2-1]+[4(v_2-1)^2-1]}{4\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{[4(v_2-1)^2-1]+[4(v_3-1)^2-1]}{4\int_{x_2}^{x_3}\sqrt{r(t)}dt} + \cdots + \frac{[4(v_{n-1}-1)^2-1]+(4(v_n-1)^2-1)}{4\int_{x_{n-1}}^{x_n}\sqrt{r(t)}dt} - \frac{1}{2}P(x_1,x_2,x_3,\dots,x_n)\Big] + O(\frac{1}{n^2}).$$

**Remark.** Note that the reader can obtain asymptotic distribution of eigenvalues in different types of (TP) by consideration of combination of the above cases.

**Acknowledgements.** The authors would like to thank Tabriz Branch, Islamic Azad University for the financial support of this research, which is based on a research project contract.

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9/6/2012