Meta Malmquist Index Based On Trade Offs Models in Data Envelopment Analysis

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Abstract: The Trade Offs approach is an advanced tool for the improvement of the discrimination of Data Envelopment Analysis (DEA) models, Meta Malmquist Index was defined by Maria Portella and et. al (2008). In this paper we compute the Meta Malmquist Index in Trade Offs model in DEA and we compare, obtaining results, of Meta Malmquist Index in different models of DEA, Variable Return to Scale (VRS), Constant Return to Scale (CRS) and Trade Offs (T-O). Numerical example is given for the purpose of illustration and we will show the management science is effective on efficiency of Decision Making Units (DMUs). The main advantage this index is that, it is circular.

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1. Introduction

Data Envelopment Analysis (DEA) is a mathematical programing technique that measures the relative efficiency of decision making units (DMUs) with multiple inputs and outputs, Charnes et. al (1978). First proposed DEA as an evaluation tool to measure and compare the relative efficiency of DMUs, their model assumed constant return to scale (CRS, the CCR model). It was developed for variable return to scale (VRS, the BCC model) by Banker et. al (1984). Podinovski suggests the incorporation of production Trade Offs in to DEA models, under this circumstance (Podinovski 2004), when we use Trade Offs in our models, the original technology expands to include the new area, Podinovski and et. al (2004) show that the production possibility set (PPS), generated by the traditional DEA axioms, may not include all the producible production points, the PPS generated by the DEA models is only the subset of the PPS with Trade Offs. Podinovski also describes the theatrical development of Trade Offs and demonstrated that Trade Offs can improve the traditional meaning of efficiency as a radial impronment factor for input or outputs (Podinovski, 2007a, 2007b). The Malmquist Index is the most important index for measuring the relative productivity change of DMUs in multiple time periods by DEA, for the first time the Malmquist Index was introduced by Caves et al (1982) for measuring the Malmquist Index. Fare et. al (1992, 1994), they computed the Malmquist index in CRS and VRS of DEA models. Also Maria Portella and Thanassoulis, defined Meta Efficiency and based on Meta Malmquist Index, they computed Meta Malmquist Index in CRS and VRS models of DEA. Meta Malmquist computes change of Meta Efficiency. The structure of the paper is as follows. In section 2 Trade Offs model of DEA is described. Section 3, we introduce Meta Malmquist Index in different models of DEA. In section 4, we explain advantage of Meta Malmquist Index and in the section 5 using the Meta frontier to compare productivities of DMUs. In section 6 we explained comparing two units at two different point in time To illustrate numerical example is brought in section 7. The last section summarizes and concludes.

2. Trade Offs in DEA models

Considering the observed output vector as $Y_j \in R^s$ and the input vector as $X_j \in R^m$, we assume that the inputs and outputs are nonnegative and $X_j \neq 0$, $Y_j \neq 0$ for DMU_j, j=1,2,…,n.

A Trade Off is a judment of possible variation in some input and or output levels, with which DMU can work without changing the other inputs and or outputs. For example, in the case of two inputs and a single output, the trade-off (P, Q) = (2,−1, 0) indicates that the DMU can work by increasing the first input by two and decreasing the second input one without changing its output (for more details, see Podinovski, 2004).

Now, suppose we have k Trade Offs. We shall represent the Trade Offs in the following form: (P_r, Q_r), where r = 1, 2 ,…, k. Also, the vector $P_r \in R^m$ and $Q_r \in R^s$ modify the inputs and outputs, respectively.For using Trade Offs in DEA models, Podinovski makes some assumptions and extends the axioms of PPS in the following manner:

Assumption:

1-All the DMUs should accept the Trade Offs.

2- Each Trade Off can be used repetitively by the DMUs.

Extended axioms:
1- (Nonempty). The observed $(X_j, Y_j) \in T$; j = 1, 2, ..., n.
2- (Proportionality). If $(X,Y) \in T$, then $(\lambda X, \lambda Y) \in T$ for all $\lambda \geq 0$.
3- (Convexity). The set T is convex.
4- (Free disposability). If $(X, Y) \in T$, $\bar{X} \geq X$, $\bar{Y} \geq Y$, then $(\bar{X}, \bar{Y}) \in T$.
5- (Feasibility of Trade Offs). Let $(X, Y) \in T$. Then for any Trade Off r in the form of $(P_r, Q_r) \in T$ and any $\pi_r \geq 0$, the unit $(X + \pi_r P_r, Y + \pi_r Q_r) \in T$, provided that $X + \pi_r P_r \geq 0$ and $Y + \pi_r Q_r \geq 0$.
6- (Closeness). The set T is closed.
7- (Minimum extrapolation). T is the smallest set that satisfies axiom 1-6. (Where T is, T = {(X, Y)| output vector Y≥ 0 can produced from input vector X ≥0}).
Now, the PPS can be defined on the basis of the following.
The minimal PPS (PPS_{TO}) that satisfies axioms (1) − (7) is:
$PPS_{TO} = \{(X, Y) | Y = \bar{Y}\lambda + \sum_{t=1}^{k} \pi_t Q_t - e, X = \bar{X}\lambda + \sum_{t=1}^{k} \pi_t P_t + d, \lambda \in R_+^n, \pi \in R_{+,}^k d \in R_+^m \text{ and } e \in R_+^s\}$, (see Podinovski (2004)).
Based on PPSTO, for assessing the relative efficiency of DMUP (p = 1, 2, ..., n) that is defined from this PPS, we have the following model:
DEA model with trade-offs technology and input orientation

$$
\begin{aligned}
&\text{Min } \theta_p \\
&\text{S.t} \quad \bar{X}\lambda + \sum_{t=1}^{k} \pi_t P_t \leq \theta_p X_p \\
&\qquad \bar{Y}\lambda + \sum_{t=1}^{k} \pi_t Q_t \geq Y_p \qquad (1) \\
&\qquad \lambda, \pi \geq 0, \ \theta_p \ \text{sign free}
\end{aligned}
$$

DEA model with trade-offs technology and output orientation

$$
\begin{aligned}
&\text{Max } \theta_p \\
&\text{S.t} \quad \bar{X}\lambda + \sum_{t=1}^{k} \pi_t P_t \leq X_p \\
&\qquad \bar{Y}\lambda + \sum_{t=1}^{k} \pi_t Q_t \geq \theta_p Y_p \qquad (2) \\
&\qquad \lambda, \pi \geq 0, \ \theta_p \ \text{sign free}
\end{aligned}
$$

Now, by considering the definition of PPS_{TO}^t, we have the following problem with different frontiers (t = 1, 2):
DEA model with Trade Offs technology and input orientation
Frontier-period=t, DMUp-period=t

$$
\begin{aligned}
&\text{Min } \theta_p^t \\
&\text{S.t} \quad \bar{X}^t\lambda^t + \sum_{t=1}^{k} \pi_r P_r^t \leq \theta_p^t X_p^t \\
&\qquad \bar{Y}^t\lambda^t + \sum_{t=1}^{k} \pi_r Q_r^t \geq Y_p^{t+1} \qquad (3) \\
&\qquad \lambda^t, \pi \geq 0, \theta_p^t \ \text{sign free}
\end{aligned}
$$

DEA model with Trade Offs technology and input orientation
Frontier-period=t + 1 DMUp-period=t + 1

$$
\begin{aligned}
&\text{Min } \theta_p^{t+1} \\
&\text{S.t} \quad \bar{X}^{t+1}\lambda^{t+1} + \sum_{t=1}^{k} \pi_r P_r^{t+1} \leq \theta_p^{t+1} X_p^{t+1} \\
&\qquad \bar{Y}^{t+1}\lambda^{t+1} + \sum_{t=1}^{k} \pi_r Q_r^{t+1} \geq Y_p^{t+1} \qquad (4) \\
&\qquad \lambda^t, \pi \geq 0, \theta_p^{t+1} \ \text{sign free}
\end{aligned}
$$

DEA model with Trade Offs technology and input orientation
Frontier-period=t DMUp-period=t + 1

$$
\begin{aligned}
&\text{Min } \theta_p^t \\
&\text{S.t} \quad \bar{X}^t\lambda^t + \sum_{t=1}^{k} \pi_r P_r^t \leq \theta_p^t X_p^{t+1} \\
&\qquad \bar{Y}^t\lambda^t + \sum_{t=1}^{k} \pi_r Q_r^t \geq Y_p^{t+1} \qquad (5) \\
&\qquad \lambda^t, \pi \geq 0, \theta_p^t \ \text{sign free}
\end{aligned}
$$

DEA model with Trade Offs technology and input orientation
Frontier-period=t + 1 DMUp-period=t

$$
\begin{aligned}
&\text{Min } \theta_p^{t+1} \\
&\text{S.t} \quad \bar{X}^{t+1}\lambda^{t+1} + \sum_{t=1}^{k} \pi_r P_r^{t+1} \leq \theta_p^{t+1} X_p^t \\
&\qquad \bar{Y}^{t+1}\lambda^{t+1} + \sum_{t=1}^{k} \pi_r Q_r^{t+1} \geq Y_p^t \qquad (6) \\
&\qquad \lambda^t, \pi \geq 0, \theta_p^{t+1} \ \text{sign free}
\end{aligned}
$$

Where X_p^t is the input vector and Y_p^t is the output vector for DMUp (p = 1, 2 ,…, n) in period t.
First, we define EEC and ETC, consider the following equations:

$$EEC = \frac{\theta_{(t+1)}^{t+1(TO)}}{\theta_{(t)}^{t(TO)}} \qquad (7) \qquad \text{and}$$

$$ETC = \left[\frac{\theta_{(t)}^{t(TO)}}{\theta_{(t)}^{t+1(TO)}} \times \frac{\theta_{(t+1)}^{t(TO)}}{\theta_{(t+1)}^{t+1(TO)}}\right]^{\frac{1}{2}} \qquad (8)$$

EEC define the changes of efficiencies for DMUp between two periods based on Trade Off technology. ETC define the change of Trade Off frontiers for DMUp between two periods.

These two definitions can present a decomposition of the EMI (Expanded Malmquist Index) as follows:

EMI = EEC × ETC (9)

Consider another equation:

REC define the changes of efficiencies for DMUp between two periods, which includes any change that come from rules and regulations based trade offs. Then,

EMI = EC×REC×ETC (10)

or EMI = PEC×SEC×REC×ETC (11)

Where

$$EC = \frac{\theta_{(t+1)}^{t+1(CRS)}}{\theta_{(t)}^{t(CRS)}} \quad (12) \quad \text{and}$$

$$TC = \left[\frac{\theta_{(t)}^{t(CRS)}}{\theta_{(t)}^{t+1(CRS)}} \times \frac{\theta_{(t+1)}^{t(CRS)}}{\theta_{(t+1)}^{t+1(CRS)}}\right]^{\frac{1}{2}} \quad (13)$$

$$PEC = \frac{\theta_{(t+1)}^{t+1(VRS)}}{\theta_{(t)}^{t(VRS)}} \quad (14) \quad \text{and}$$

$$SEC = \left[\frac{\theta_{(t)}^{t(VRS)}}{\theta_{(t)}^{t+1(CRS)}} \times \frac{\theta_{(t+1)}^{t(CRS)}}{\theta_{(t+1)}^{t+1(VRS)}}\right]^{\frac{1}{2}} \quad (15)$$

So we have

Malmquist Index = EC×TC (16)

or Malmquist Index = PEC×SEC×TC (17)

We now, DEA models for measuring efficiency of DMUp(p = 1, 2, ..., n) with problems below:

DEA model with CRS technology and input orientation

$$\begin{aligned} &\text{Min } \theta_p \\ &\text{S.t} \quad \bar{X}\lambda \le \theta_p X_p \\ &\qquad \bar{Y}\lambda \ge Y_p \qquad\qquad (18) \\ &\qquad \lambda \ge 0,\ \theta_p \text{ sign free} \end{aligned}$$

DEA model with VRS technology and input orientation

$$\begin{aligned} &\text{Min } \theta_p \\ &\text{S.t} \quad \bar{X}\lambda \le \theta_p X_p \\ &\qquad \bar{Y}\lambda \ge Y_p \qquad\qquad (19) \\ &\qquad 1.\lambda = 1 \\ &\qquad \lambda \ge 0,\ \theta_p \text{ sign free} \end{aligned}$$

Thus, if $\theta_{j(t)}^{T}$ is the efficiency measure of unit j observed at time t relative to the technology boundary of time period T, then a Malmquist Index of the change of its productivity between period t and t + 1 is given by Malmquist productivity change index j equivalent $\frac{\theta_{j(t+1)}^{T}}{\theta_{j(t)}^{T}}$ (20) and the traditional Malmquist index is computed by $MI = \frac{\theta_{j(t+1)}^{T+1}}{\theta_{j(t)}^{T}} \times \left[\frac{\theta_{j(t+1)}^{T}}{\theta_{j(t+1)}^{T+1}} \times \frac{\theta_{j(t)}^{T}}{\theta_{j(t)}^{T+1}}\right]^{\frac{1}{2}}$ (21) where $\frac{\theta_{j(t+1)}^{T+1}}{\theta_{j(t)}^{T}}$ (22) is an efficiency change and $\left[\frac{\theta_{j(t+1)}^{T}}{\theta_{j(t+1)}^{T+1}} \times \frac{\theta_{j(t)}^{T}}{\theta_{j(t)}^{T+1}}\right]^{\frac{1}{2}}$ (23) is a technological change or boundary shift.

$$EC = \frac{\theta_{j(t+1)}^{T+1(CRS)}}{\theta_{j(t)}^{T(CRS)}} \quad (24)$$

$$PEC = \frac{\theta_{j(t+1)}^{T+1(VRS)}}{\theta_{j(t)}^{T(VRS)}} \quad (25)$$

$$TC = \left[\frac{\theta_{j(t)}^{T(CRS)}}{\theta_{j(t)}^{T+1(CRS)}} \times \frac{\theta_{j(t+1)}^{T(CRS)}}{\theta_{j(t+1)}^{T+1(CRS)}}\right]^{\frac{1}{2}} \quad (26)$$

$$SEC = \left[\frac{\theta_{j(t)}^{T(VRS)}}{\theta_{j(t)}^{t(CRS)}} \times \frac{\theta_{j(t+1)}^{T+1(CRS)}}{\theta_{j(t+1)}^{T+1(VRS)}}\right]^{\frac{1}{2}} \quad (27)$$

$$EEC = \frac{\theta_{j(t+1)}^{T+1(TO)}}{\theta_{j(t)}^{T(TO)}} \quad (28)$$

$$ETC = \left[\frac{\theta_{j(t)}^{T(TO)}}{\theta_{j(t)}^{T+1(TO)}} \times \frac{\theta_{j(t+1)}^{T(TO)}}{\theta_{j(t+1)}^{T+1(TO)}}\right]^{\frac{1}{2}} \quad (29)$$

MI = EC×TC (30)

MI = PEC×SEC×TC (31)

MI = EEC×ETC (32)

EMI = EC ×REC ×ETC (33)

EMI = PEC ×SEC ×REC ×ETC (34)

3. Meta Malmquist Index

Consider DMUs (1, 2, ..., n) observed over time period t, t = 1, 2, ..., T so that meta period covers T periods. Let (X_{ij}^{t}, Y_{rj}^{t}) be respectively the ith input and rth output level of DMUj in period t within the meta period, The meta efficiency of DMUj0 and j0 ∈ 2 (1, 2, ..., n) observed in some period τ ∈

$(1,2,\dots,T)$ is $\theta^{m}_{j0(\tau)}$, where $\theta^{m}_{j0(\tau)}$, is the optimal value of k_{j0} in model (35) below:

$$\theta^{m}_{j0(\tau)} = \min k_{j0}$$
$$\text{S.t} \quad \sum_{t=1}^{T}\sum_{j=1}^{n}\lambda_{jt}\, x^{t}_{ij} \le k_{j0}x^{\tau}_{ij0} \quad i = 1,2,\dots,m$$
$$\sum_{t=1}^{T}\sum_{j=1}^{n}\lambda_{jt}\, y^{t}_{rj} \ge y^{\tau}_{rj0} \quad r = 1,2,\dots,s \qquad (35)$$
$$\lambda_{jt} \ge 0 \ , j = 1,2,\dots,n \qquad k_{j0} \quad \text{free}$$

Model (35) relates to constant return to scale technologies and has input orientation. Now let θ^{m}_{jt} be the Meta efficiency of unit j as observed in period t and computed using a model such as (35). Then we have:

Meta efficiency of unit j observed in period t = within period t efficiency of unit j ×technological gap between period t boundary and the Meta frontier.

Putting the foregoing decomposition in symbols we have $\theta^{m}_{jt} = \theta^{T}_{jt} \times TG_{jt}$ where θ^{T}_{jt} is obtained for each unit j0 as the optimal value of k_{j0} in (35) after dropping all instances apart from those occurring in period t, and TG_{jt} is retrieved residually as $TG_{jt} = \frac{\theta^{m(CRS)}_{jt}}{\theta^{T(CRS)}_{jt}}$ thus $\theta^{m(CRS)}_{jt} = \theta^{T(CRS)}_{jt} \times TG^{CRS}_{jt}$ (36)

In this paper, we obtain θ^{m}_{jt} by solving Trade Offs model. First, we introduce Meta Malmquist Index for DMUj between period t and t + 1.

Meta Malmquist Index=$MI^{j}_{t,t+1}=\frac{\theta^{m}_{j(t+1)}}{\theta^{m}_{jt}}$ (Maria Portela and Thanassoulis(2008)). So

$$MI^{j(CRS)}_{t,t+1} = \frac{\theta^{m}_{j(t+1)}}{\theta^{m}_{jt}} = \frac{\theta^{T+1(CRS)}_{j(t+1)}}{\theta^{T(CRS)}_{j(t)}} \times \frac{TG^{CRS}_{j(t+1)}}{TG^{CRS}_{j(t)}} \qquad (37)$$

The term $\frac{\theta^{T+1}_{j(t+1)}}{\theta^{T}_{j(t)}}$ captures the efficiency change of unit j from year t + 1 as in traditional Malmquist Index of productivity change.

The term $\frac{TG_{j(t+1)}}{TG_{j(t)}}$ captures frontier shift between period t and t+1, $\frac{TG_{j(t+1)}}{TG_{j(t)}} = \frac{\theta^{m}_{j(t+1)}}{\theta^{T+1}_{j(t+1)}} \Big/ \frac{\theta^{m}_{j(t)}}{\theta^{T}_{j(t)}}$.

For computing VRS efficiency scores all that is to add to DEA models (such as (35)) the convexity constraint imposing the sum of all lambdas to be 1, $\sum_{t=1}^{T}\sum_{j=1}^{n}\lambda_{jt} = 1$, which in the case of (35). Would yield the Meta efficiency VRS score of unit j0. Now, we write DEA model with Trade Offs technology and input orientation. Suppose we have k Trade Offs, and we show (P_h, Q_h) where h = 1, 2, ..., k, then the meta efficiency of DMU_{j0} and j0 ∈ (1, 2, ..., n) observed in some period τ, $\tau \in (1,2,\dots,T)$ is $\theta^{m}_{j0(\tau)}$, where $\theta^{m}_{j0(\tau)}$ is the optimal value of k_{j0} in model (38) below:

$$\theta^{m}_{j0(\tau)} = \min k_{j0}$$
$$\text{S.t} \quad \sum_{t=1}^{T}\Big(\sum_{j=1}^{n}\lambda_{jt}\, x^{t}_{ij} + \sum_{h=1}^{k}\pi_{ht}\, P^{t}_{ih}\Big) \le k_{j0}x^{\tau}_{ij0} \quad i = 1,2,\dots,m$$
$$\sum_{t=1}^{T}\Big(\sum_{j=1}^{n}\lambda_{jt}\, y^{t}_{rj} + \sum_{h=1}^{k}\pi_{ht}\, Q^{t}_{rh} \ge y^{\tau}_{rj0} \quad r = 1,2,\dots,s \qquad (38)$$
$$\lambda_{jt} \ge 0 \ , j = 1,2,\dots,n \qquad k_{j0} \quad \text{free} \quad \pi_{ht} \ge 0 \quad h = 1,2,\dots k$$

Model (38) relates to Trade Offs technologies and has an input orientation. Thus $\theta^{m(TO)}_{jt}$ is the Meta efficiency of unit j as observed in period t and computed using a model such as (38). $\theta^{m(TO)}_{jt} = \theta^{T(TO)}_{jt} \times TG^{TO}_{jt}$ is obtained for each unit j0 as the optimal value of k_{j0} in (38) after dropping all instances apart from those occurring in period t, then $TG^{TO}_{jt} = \frac{\theta^{m}_{jt}}{\theta^{T(TO)}_{jt}}$ and we obtain $\theta^{T(TO)}_{j0\tau}$ from solving this program:

$$\theta^{T}_{j0(\tau)} = \min k_{j0}$$
$$\text{S.t} \quad \sum_{j=1}^{n}\lambda_{jt}\, x^{T}_{ij} + \sum_{h=1}^{k}\pi_{ht}\, P^{T}_{ih} \le k_{j0}x^{\tau}_{ij0} \quad i = 1,2,\dots,m$$
$$\sum_{j=1}^{n}\lambda_{jt}\, y^{T}_{rj} + \sum_{h=1}^{k}\pi_{ht}\, Q^{T}_{rh} \ge y^{\tau}_{rj0} \quad r = 1,2,\dots,s \qquad (39)$$
$$\lambda_{jt} \ge 0 \ , j = 1,2,\dots,n \qquad k_{j0} \quad \text{free} \quad \pi_{ht} \ge 0 \quad h = 1,2,\dots k$$

Then
$$MI^{j(TO)}_{t,t+1} = \frac{\theta^{m(TO)}_{j(t+1)}}{\theta^{m(TO)}_{jt}} = \frac{\theta^{T+1(TO)}_{j(t+1)}}{\theta^{T(TO)}_{j(t)}} \times \frac{TG^{TO}_{j(t+1)}}{TG^{TO}_{j(t)}} \qquad (40)$$

Therefor
$$\frac{TG^{TO}_{j(t+1)}}{TG^{TO}_{j(t)}} = \frac{\theta^{m(TO)}_{j(t+1)}}{\theta^{m(TO)}_{jt}} \times \frac{\theta^{T(TO)}_{j(t)}}{\theta^{T+1(TO)}_{j(t+1)}} \qquad (41)$$

$$MI^{j(TO)}_{t,t+1} = EEC \times \frac{TG^{TO}_{j(t+1)}}{TG^{TO}_{j(t)}} = EC \times REC \times \frac{TG^{TO}_{j(t+1)}}{TG^{TO}_{j(t)}} = PEC \times SEC \times REC \times \frac{TG^{TO}_{j(t+1)}}{TG^{TO}_{j(t)}} \qquad (42)$$

Thus
$$MI^{j(TO)}_{t,t+1} = \frac{EMI}{ETC} \times \frac{TG^{TO}_{j(t+1)}}{TG^{TO}_{j(t)}} \qquad (43)$$

4. Advantage of the Circular Meta Malmquist Index

To see that note that $\frac{\theta^m_{j(t+2)}}{\theta^m_{jt}} = \frac{\theta^m_{j(t+2)}}{\theta^m_{j(t+1)}} \times \frac{\theta^m_{j(t+1)}}{\theta^m_{j(t)}}$. That is the productivity change between period t and t+2 is the product of the successive productivity change from period t to t + 1 and from period t + 1 to t + 2.

$$\frac{\theta^{m(TO)}_{j(t+2)}}{\theta^{m(TO)}_{jt}} = \frac{\theta^{m(TO)}_{j(t+2)}}{\theta^{m(TO)}_{j(t+1)}} \times \frac{\theta^{m(TO)}_{j(t+1)}}{\theta^{m(TO)}_{j(t)}} \quad (44)$$

Similarly, we will want $\frac{\theta^{T+2(TO)}_{j(t+2)}}{\theta^{T(TO)}_{jt}} = \frac{\theta^{T+2(TO)}_{j(t+2)}}{\theta^{T+1(TO)}_{j(t+1)}} \times \frac{\theta^{T+1(TO)}_{j(t+1)}}{\theta^{T(TO)}_{j(t)}}$ is an efficiency change from period t to t + 2 and $\frac{TG^{T+2(TO)}_{j(t+2)}}{TG^{T(TO)}_{jt}} = \frac{TG^{T+2(TO)}_{j(t+2)}}{TG^{T+1(TO)}_{j(t+1)}} \times \frac{TG^{T+1(TO)}_{j(t+1)}}{TG^{T(TO)}_{j(t)}}$ is a boundary shift from period t to t + 2. And meta efficiency VRS score of unit j0 is $\theta^{m(TO)}_{j(t)} = \theta^{T(VRS)}_{j(t)} \times \frac{\theta^{m(VRS)}_{j(t)}}{\theta^{T(VRS)}_{jt}} \times \frac{\theta^{m(TO)}_{j(t)}}{\theta^{m(VRS)}_{jt}} = \theta^{T(VRS)}_{j(t)} \times TGV_{jt} \times MSET_{jt}$ that is meta efficiency decomposes in to within period efficiency in relation to a VRS frontier $\theta^{T(VRS)}_{j(t)}$, technological gap between the VRS frontier in t and the VRS meta frontier $\frac{\theta^{m(VRS)}_{j(t)}}{\theta^{T(VRS)}_{jt}}$, labeled TGV and meta scale efficiency $\frac{\theta^{m(TO)}_{j(t)}}{\theta^{m(VRS)}_{jt}}$ labeled MSET. Note that MSET capture the distance between the TO and V RS meta frontiers, at the input output mix of unit j as observed in period t, thus the circular Meta Malmquist Index, defined asn $MI^{j(TO)}_{t,t+1} = \frac{\theta^{m(TO)}_{j(t+1)}}{\theta^{m(TO)}_{jt}}$ can be decomposed as shown:

$$MI^{j(TO)}_{t,t+1} = \frac{\theta^{T+1(VRS)}_{j(t+1)}}{\theta^{T(VRS)}_{jt}} \times \frac{TGV_{j(t+1)}}{TGV_{j(t)}} \times \frac{MSET_{j(t+1)}}{MSET_{j(t)}} \quad (45)$$

$$MI^{j(TO)}_{t,t+1} = PEC \times \frac{TGV_{j(t+1)}}{TGV_{j(t)}} \times \frac{MSET_{j(t+1)}}{MSET_{j(t)}} \quad (46)$$

Pure technical efficiency change $\frac{\theta^{T+1(VRS)}_{j(t+1)}}{\theta^{T(VRS)}_{jt}}$, frontier shift between VRS frontiers $\frac{TGV_{j(t+1)}}{TGV_{j(t)}}$ and Meta scale efficiency change $\frac{MSET_{j(t+1)}}{MSET_{j(t)}}$.

5. Using the meta frontier to compare productivities of units

Using now the unit specific boundaries we can compute two efficiency scores for each unit instance jt (i.e unit j observed in year t). One efficiency will be relative to the meta frontier as before and denoted θ^m_{jt} , while the second will be relative to the unit specific boundary as defined above and it is denoted $\theta^{U_j}_{jt}$, where the index U_j relates to the unit specific boundary of unit j, and $\theta^{U_j}_{jt}$ in relation to unit j0 ∈ (1, 2, ..., n) observed in period τ ∈(1, 2, ..., T) is obtained buy solving model (49) (M. Portela and et.al (2008)).

$$\begin{aligned} &\theta^{U_j}_{j0(\tau)} = min\ k_{j0} \\ &\text{S.t} \quad \textstyle\sum_{t=1}^{T} \lambda_{j0t}\, x^t_{ij0} \le k_{j0} x^\tau_{ij0} \qquad i = 1,2,\dots,m \\ &\qquad \textstyle\sum_{t=1}^{T} \lambda_{j0t}\, y^t_{rj0} \ge y^\tau_{rj0} \qquad r = 1,2,\dots,s \qquad (49) \\ &\qquad \lambda_{jt} \ge 0 \quad , j = 1,2,..,n \qquad k_{j0} \quad free \end{aligned}$$

Notation in (49) is as in (35). Note now that we have $\theta^m_{jt} = \theta^{U_j}_{jt} \times UG_{jt}$ where is retrieved residually and it measure the distance from the unit specific frontier to the Meta frontier, we shall refer to UG_{jt} as the unit frontier gap for unit j, measured at the units input-output mix in time period t.

Now we will obtain $\theta^{U_j}_{jt}$ in Trade Offs model in DEA, therefore we will have $\theta^{U_j}_{jt}$ by solving model (50). Suppose we have k Trade Offs, and we show (P_h, Q_h) where h = 1, 2, ..., k, then DMU_{j0} and j0 ∈ (1, 2, ..., n) observed in some period τ, $\tau \in (1,2,..,T)$ is $\theta^m_{j0(\tau)}$ where $\theta^m_{j0(\tau)}$ is the optimal value of k_{j0} in model (50) below:

$$\begin{aligned} &\theta^{U_j}_{j0(\tau)} = min\ k_{j0} \\ &\text{S.t} \quad \textstyle\sum_{t=1}^{T} (\lambda_{j0t}\ x^t_{ij0} + \sum_{h=1}^{k} \pi_{ht}\ P^T_{ih}) \le k_{j0} x^\tau_{ij0} \qquad i = 1,2,\dots,m \\ &\qquad \textstyle\sum_{t=1}^{T} (\lambda_{j0t}\ t + \sum_{h=1}^{k} \pi_{ht}\ Q^T_{rh}\ ) \ge y^\tau_{rj0} \qquad r = 1,2,\dots,s \qquad (50) \\ &\lambda_{j0t} \ge 0 \quad , j = 1,2,..,n \qquad k_{j0} \quad free \quad \pi_{ht} \ge 0 \quad h = 1,2,\dots k \end{aligned}$$

The $\theta^m_{jt} = \theta^{U_j(TO)}_{jt} \times UG^{TO}_{jt}$ where UG^{TO}_{jt} is retrieved residually and it measures the distance from unit specific frontier to the Meta frontier. We shall refer to UG_{jt} as the unit frontier gap for unit j, measured at the units input-output mix in time period t. We can now the Meta efficiency of two units at the same period in time to compare their productivities. Let us consider unit j and k and let us take their instances in period t. A measure of their relative productivity is given by the ratio of their Meta efficiencies, $MI_{kj}^{t}=\frac{\theta_{jt}^{m}}{\theta_{kt}^{m}}$. When MI_{kj}^{t} is greater than 1 it means that productivity at unit j is higher than that of unit k in period t. Values lower than 1 mean the converse.

By Using this definition which decomposes Meta efficiency in to within unit and unit frontier gap we can decompose the index of comparative unit productivity MI_{kj}^{t} of unit j and k as observed in period t as follows (Maria Portela and Thanassoulis (2008)):

$$MI_{kj}^{t}=\frac{\theta_{jt}^{m}}{\theta_{kt}^{m}}=\frac{\theta_{jt}^{U_j}}{\theta_{kt}^{U_k}}\times\frac{UG_{jt}}{UG_{kt}} \qquad (52)$$

The term $\frac{\theta_{jta}^{U_j}}{\theta_{kt}^{U_k}}$ captures the component of the relative productivity of units j and k accounted for by the distance of unit j in period t from its unit specific boundary U_j compared to the corresponding distance of unit k in period t from its own unit specific boundary U_k. This term will be referred to the general as within unit efficiency difference between j and k at time t. The term $\frac{UG_{jt}}{UG_{kt}}$ captures the component of the relative productivity of units j and k at time t accounted for by the distance of the unit specific boundary of unit j from the meta frontier, taken at the input-output mix of unit j at time t, compared to the corresponding distance of unit specific boundary of unit k taken at its input-output mix in period t. Given that the Meta frontier is stationary the ratio in question reflects the distance between the unit specific boundaries taken at their respective input-output mixes in period t. Thus the term $\frac{UG_{jt}}{UG_{kt}}$ is analogous to the Frontier Shift component in the Meta Malmquist Index of productivity change over time as defined in

$$MI_{t,t+1}^{j}=\frac{\theta_{j(t+1)}^{m}}{\theta_{jt}^{m}}=\frac{\theta_{j(t+1)}^{T+1}}{\theta_{jt}^{T}}\times\frac{TG_{j(t+1)}}{TG_{j(t)}} \qquad (53)$$

But here the frontier shift is not over time but rather between production units. So we shall refer to the term $\frac{UG_{jt}}{UG_{kt}}$ as unit frontier shift between k and j at time t. Similarly, suppose we have k Trade Offs then we will observe:

$$MI_{kj}^{t(TO)}=\frac{\theta_{jt}^{m(TO)}}{\theta_{kt}^{m(TO)}}=\frac{\theta_{jt}^{U_j(TO)}}{\theta_{kt}^{U_k(TO)}}\times\frac{UG_{jt}^{(TO)}}{UG_{kt}^{(TO)}} \qquad (54)$$

6. Comparing two units at two different point in time

We know generalize the above concepts to compare and decompose the productivities of two units at two different points in time. Let us consider units j and k and let us take their instance in period s and t respectively. A measure of the relative productivity is given by the ratio of their meta efficiencies, $MI_{kj}^{ts}=\frac{\theta_{js}^{m}}{\theta_{kt}^{m}}$ and we will have (for more details seeMaria Portela and Thanassoulis (2008)):

$$MI_{kj}^{ts}=\frac{\theta_{js}^{m}}{\theta_{kt}^{m}}=\frac{\theta_{js}^{m}}{\theta_{jt}^{m}}\times\frac{\theta_{jt}^{m}}{\theta_{kt}^{m}} \qquad (55)$$

That is MI_{kj}^{ts} decomposes in two indices, The first index is $\frac{\theta_{js}^{m}}{\theta_{jt}^{m}}$, This will be referred to as productivity change over time. Here it captures the change in the productivity of unit j between period t and s. The second index is $\frac{\theta_{jt}^{m}}{\theta_{kt}^{m}}$ this will be referred to as productivity difference between contemporaneous units. Here it captures the difference in the productivity of units j and k in period t Note that a similar decomposition to that in MI_{kj}^{ts} , but using unit k rather than j as a reference is also possible. That is we could have:

$$MI_{kj}^{ts}=\frac{\theta_{js}^{m}}{\theta_{kt}^{m}}=\frac{\theta_{ks}^{m}}{\theta_{kt}^{m}}\times\frac{\theta_{js}^{m}}{\theta_{ks}^{m}} \qquad (56)$$

By using all of formulation in this paper we will have:

$$MI_{kj}^{ts}=\frac{\theta_{js}^{S}}{\theta_{kt}^{T}}\times\frac{TG_{js}}{TG_{jt}}\times\frac{\theta_{jt}^{U_j}}{\theta_{kt}^{U_k}}\times\frac{UG_{jt}}{UG_{kt}} \qquad (57)$$

That is the difference in productivity between units k and j observed in period t and s decomposes in to efficiency change $\frac{\theta_{js}^{S}}{\theta_{kt}^{T}}$ for unit j between periods t and s, period frontier shift $\frac{TG_{js}}{TG_{jt}}$ between s and t, at the input-output mix of unit j, within unit efficiency difference $\frac{\theta_{jt}^{U_j}}{\theta_{kt}^{U_k}}$ between j and k at time t and unit frontier shift $\frac{UG_{jt}}{UG_{kt}}$between units j and k at their input-output mix in period t. Similarly, with having k Trade Offs we will have:

$$MI_{kj}^{ts(TO)}=\frac{\theta_{js}^{m(TO)}}{\theta_{kt}^{m(TO)}}=\frac{\theta_{ks}^{m(TO)}}{\theta_{kt}^{m(TO)}}\times\frac{\theta_{js}^{m(TO)}}{\theta_{ks}^{m(TO)}} \quad (58)$$

$$MI_{kj}^{ts} = \frac{\theta_{js}^{S(TO)}}{\theta_{kt}^{T(TO)}}\times\frac{TG_{js}{}^{TO}}{TG_{jt}{}^{TO}}\times\frac{\theta_{jt}^{U_j(TO)}}{\theta_{kt}^{U_k(TO)}}\times\frac{UG_{jt}{}^{TO}}{UG_{kt}{}^{TO}} \quad (59)$$

7. Example

Example 1: Consider Table (1)... (6) in this Tables, we have ten DMUs with three inputs and five outputs at three periods. Data have been taken from a commercial Bank in IRAN for seven branches. Assume that all DMUs agree as being true for the following judgments at two periods (we have three Trade Offs in each period).

1.$(P_1^t, Q_1^t)= (P_1^{t+1}, Q_1^{t+1}) =(P_1^{t+2}, Q_1^{t+2}) =(p_{11}^t, p_{12}^t, p_{13}^t, q_{11}^t, q_{12}^t, q_{13}^t, q_{14}^t, q_{15}^t)=$
$(-10,10000000,1000000000,100000000,10000000000,1000000000,1100000000,10000000000)$
2 .$(P_2^t, Q_2^t)= (P_2^{t+1}, Q_2^{t+1}) =(P_2^{t+2}, Q_2^{t+2}) =(p_{21}^t, p_{22}^t, p_{23}^t, q_{21}^t, q_{22}^t, q_{23}^t, q_{24}^t, q_{25}^t)=$
$(-14, 20000000, 2000000000, 11100000000, 10000000000, 100000000000, 9000000000, 900000000000)$
3. $(P_3^t, Q_3^t)= (P_3^{t+1}, Q_3^{t+1}) =(P_3^{t+2}, Q_3^{t+2}) =(p_{31}^t, p_{32}^t, p_{33}^t, q_{31}^t, q_{32}^t, q_{33}^t, q_{34}^t, q_{35}^t)$=(0,0,0,0,0,0,0,0)
$(-10, 10000000, 2000000000, 100000000000, 120000000000, 100000000000, 80000000000,-30000000)$

Table (1), Inputs in period$t$

Unit	x_1	x_2	x_3
DMU_1	24.6	1163006699	21214797126
DMU_2	26.51	433148930	586933516
DMU_3	14.69	2338234805	2423898801
DMU_4	20.15	1344914230	4260524118
DMU_5	19.74	488806775	3658263944
DMU_6	16.32	1506571113	1700726564
DMU_7	16.3	575541160	4931297789
DMU_8	38.64	687401632	9892984883
DMU_9	30.54	1052621363	10850597644
DMU_{10}	20.04	620789783	4005818426

Table (2), Inputs in period$t+1$

Unit	x_1	x_2	x_3
DMU_1	24.6	2195224416	20705682078
DMU_2	26.51	859703238	573439627
DMU_3	14.69	4686245937	2419191001
DMU_4	20.15	2716258762	5708905141
DMU_5	19.74	965343887	3510322663
DMU_6	16.32	2957622923	1767161564
DMU_7	16.3	1169632952	5089852841
DMU_8	38.64	1380459532	9688488055
DMU_9	30.54	2156060329	11187719906
DMU_{10}	20.04	1257857335	4003741426

Table (3), Inputs in period $t+2$

Unit	x_1	x_2	x_3
DMU_1	24.6	3161147138	17899623576
DMU_2	26.51	1268741030	1310517912
DMU_3	14.69	7030453173	2270066109
DMU_4	20.15	4110060766	5708955141
DMU_5	19.74	1437235127	3509804663
DMU_6	16.32	4396171175	1823653154
DMU_7	16.3	1794807344	5284876078
DMU_8	38.64	2076989680	10217822041
DMU_9	30.54	3307801816	11216072885
DMU_{10}	20.04	1892438197	2509333739

Table (4), Outputs in period t.

Unit	y_1	y_2	y_3	y_4	y_5
DMU_1	242241992696	149927581688	739255995	137914144	428667943
DMU_2	107238188475	86355903094	9516959	230075940	80728098
DMU_3	277898251065	219123202540	270049146	19573201	532701000
DMU_4	88180358515	173092885271	74637180	625952459	5860810117
DMU_5	128834133115	88264150018	235995190	906927396	7805587173
DMU_6	298359497990	184328871243	1712460544	1213573293	11308531095
DMU_7	85714984119	125413342055	54632177	385747036	5669053062
DMU_8	135956919734	119648993075	29280821	102535073	5155800956
DMU_9	333510866500	137294730996	79558096	112179147	666055008
DMU_{10}	234535258794	116487806490	85936562	152512990	453380535

Table (5), Outputs in period $t+1$.

Unit	y_1	y_2	y_3	y_4	y_5
DMU_1	249623342159	117399292537	1540543984	178683273	428667943
DMU_2	112562861469	86202028391	25345126	345083696	84628098
DMU_3	276955477174	195761787737	466977957	48703328	537701000
DMU_4	89769875594	175034650015	85423893	711106298	6140099086
DMU_5	126452729455	81056170548	1021865861	1097224634	7975379183
DMU_6	313515525726	180595350698	4678292867	1420994153	12549446009
DMU_7	88201054771	115220570013	59465117	737973247	5788081726
DMU_8	146533627624	113094425933	396985110	168979756	5150091667
DMU_9	328457006339	153646949485	160072278	180257561	656055008
DMU_{10}	239480060054	119637603603	153267220	232276956	468380535

Table (6), Outputs in period $t+2$.

Unit	y_1	y_2	y_3	y_4	y_5
DMU_1	293867028198	140441085515	1761926627	219200982	417867943
DMU_2	115139451138	87285197519	24844888	435539236	165778098
DMU_3	278333223405	206467052805	791923009	64527873	537701000
DMU_4	93698153403	174197607935	320187997	767855712	5878898037
DMU_5	141765332119	91751175208	1144343535	1405955598	8089357778
DMU_6	318100978977	188236536110	6433265613	1601266241	12025630507
DMU_7	90693221404	131524255400	141621574	906022389	6897553226
DMU_8	145024864957	117506091612	636944528	234797904	5138375112
DMU_9	347376530595	156528383470	167954127	203958539	661772204
DMU_{10}	242826874768	114773321947	208500176	268924230	368380535

Table (7), Meta Efficiency and Meta Malmquist Index for $DMUs$ in CRS models of DEA.

Unit	$\theta_{jt}^{m(CRS)}$	$\theta_{jt+1}^{m(CRS)}$	$\theta_{jt+2}^{m(CRS)}$	$MI_{t,t+1}^{j(CRS)}$	$MI_{t+1,t+2}^{j(CRS)}$	$MI_{t,t+2}^{j(CRS)}$
DMU_1	0.8494	0.5976	0.6402	0.7036	1.0712	0.7537
DMU_2	1.0000	1.0000	0.5536	1.0000	0.5536	0.5536
DMU_3	1.0000	0.9844	0.9946	0.9844	1.0103	0.9946
DMU_4	0.8875	0.6896	0.6726	0.7771	0.9753	0.7579
DMU_5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
DMU_6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
DMU_7	1.0000	0.7384	0.7228	0.7384	0.9788	0.7228
DMU_8	0.8586	0.4812	0.3739	0.5604	0.7771	0.4355
DMU_9	0.9042	0.6742	0.6165	0.7456	0.9145	0.6819
DMU_{10}	1.0000	0.7907	0.6620	0.7907	0.8372	0.6620

Table (8), Meta Efficiency and Meta Malmquist Index for DMUs in Trade Offs models of DEA.

Unit	$\theta_{jt}^{m(TO)}$	$\theta_{jt+1}^{m(TO)}$	$\theta_{jt+2}^{m(TO)}$	$MI_{t,t+1}^{j(TO)}$	$MI_{t+1,t+2}^{j(TO)}$	$MI_{t,t+2}^{j(TO)}$
DMU_1	0.1935	0.1779	0.2130	0.9196	1.1971	1.1009
DMU_2	1.0000	1.0000	0.5536	1.0000	0.5536	0.5536
DMU_3	0.9352	0.8368	0.9278	0.8948	1.1088	0.9921
DMU_4	0.4920	0.3534	0.3457	0.7183	0.9781	0.7026
DMU_5	0.4835	0.3891	0.3667	0.8048	0.9423	0.7584
DMU_6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
DMU_7	0.3490	0.2920	0.3063	0.8366	1.0489	0.8775
DMU_8	0.2237	0.2035	0.1728	0.9097	0.8495	0.7727
DMU_9	0.4612	0.3662	0.3314	0.7940	0.9050	0.7185
DMU_{10}	0.7609	0.6124	0.6160	0.8049	1.0057	0.8095

8. Conclusion

Trade Off technology is used o evaluate in Malmquist, Expanded Malmquist and Meta Malmquist productivity the validity of models is shown by numerical example and a set of data from a commercial bank is used and the result from the point board of directory is quite satisfactory.

The above mation models may be extended for multicative models and the models for non redial efficiency also the Malmquist and Expanded meta Malmquist may be used for cost efficiency and revenue efficiency.

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