Application of Optimal Homotopy Asymptotic Method to the Equal Width Wave and Burger Equations

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Abstract: In this paper, we use the optimal homotopy asymptotic method (OHAM) for approximate solutions of the Equal Width Wave equations (EWW) and Burger equations, respectively. For (EWW) equations the numerical results obtained are compared with the results obtained by using variational iteration method (VIM) and Adomian decomposition method (ADM), while for burger equation comparison is made with ADM. From the obtained results it is observed that the suggested method is explicit, effective, and very easy to use. [Islam S, Nawaz R, Arif M and Shah SIA , **Application of Optimal Homotopy Asymptotic Method to the Equal Width Wave and Burger Equations.** *Life Sci J* 2012;9(4):2380-2386] (ISSN:1097-8135). http://www.lifesciencesite.com. 353

Keywords: Optimal Homotopy Asymptotic Method, Burger Equation, Equal Width Wave Equation

1. Introduction

Mathematical modeling of different problems in various fields has been discussed using formal approaches [20-32], fuzzy logic [19] and Differential equations [1-3]. Differential equations play a vital role in modeling different problems in physics, biology, chemical reactions and in engineering sciences. The one dimension non-linear differential equation, which is similar to the one dimension Navier-Stokes equation without the stress term, was presented for the first time in a paper in 1940 from Burger. It is the model for the solution of Navier-Stokes equation and is applied to laminar and turbulence flows as well. The first theoretical solution of Burger equation was given by Cole [13] which is based on Fourier series analysis using the appropriate initial and boundary conditions. Another theoretical solution was given by Madsen and Sincovec [14], based on the ”test and trial” method, using the appropriate initial and boundary conditions. The Burger equation can be used as a model for different problems of a fluid flow nature, where shocks or viscous dissipation is a major factor. It can be used as a model for any nonlinear wave transmission problem subject to dissipation [11-15].

The Equal Width Wave (EW) equation was suggested by Morrison et al., to use as a modelfor the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes [17].

Marinca and Herisanuet al. introduced a new semi analytic method (OHAM) for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. They used OHAM for understanding the behavior of nonlinear mechanical vibration of an electrical machine. By using this method they investigated solution of nonlinear equations arising in the study of fourth grade fluid past a porous plate. The convergence criterion of proposed method is similar to that of HAM and HPM, but this method is more efficient and flexible [1-3]. The proposed method has been used by many researchers for solution of Ordinary and Partial Differential equations [5-10]. The objective of this paper is to show the effectiveness of OHAM for (EWW) and Burger equations. We consider the EWW equation, derived for long waves propagating in the positive x-direction which has the form

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^3 u(x,t)}{\partial x^2\partial t}=0 \qquad (1)$$

And Burger equation of the form

$$\frac{\partial u(x,t)}{\partial t}+\alpha u^{\delta}(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}=0 \;..(2)$$

With initial condition given by

$$u(x,0)=0.5+0.5\tanh\left(\frac{-\alpha\delta}{2(\delta+1)}x\right) \qquad (3)$$

for all $0 \le x \le 1$and $t \ge 0.$ α and δ are parameters and $\delta > 0.$

The whole paper contains four sections. Each of them is analyzed as follows. The first section is the introduction. The fundamental theory of OHAM is given in the section 2.

In section 3 numerical solution of (EWW) equation is presented by OHAM and absolute errors are compared with NIM and ADM. In section 4 Comparisons are made between the results of the proposed method and ADM forBurger equation. In all cases the proposed method yields very encouraging results.

Here we start by describing the basic idea of OHAM, Consider the partial differential equation of the form:

$$\mathcal{L}\left(u\left(x,t\right)\right)+\mathcal{N}\left(u\left(x,t\right)\right)+g\left(x,t\right)=0,$$
$$x\in\Omega \qquad (4)$$
$$\mathcal{B}\left(u,\partial u/\partial t\right)=0, \qquad (5)$$

where $\mathcal{L}$ is a linear operator and $\mathcal{N}$ is nonlinear operator. $\mathcal{B}$ is boundary operator, $u\left(x,t\right)$ is an unknown function, and x and t denote spatial and time variables, respectively, Ω is the problem domain and $g\left(x,t\right)$ is a known function.

According to the basic idea of OHAM, one can construct the optimal homotopy $\psi\left(x,t;q\right):\ \Omega\times\left[0,1\right]\rightarrow R$ which satisfies

$$(1-q)\left\{\mathcal{L}\left(\psi\left(x,t;q\right)\right)+g(x,t)\right\}=\mathcal{H}(q)\left\{\begin{array}{l}\mathcal{L}\left(\psi\left(x,t;q\right)\right)+\\ \mathcal{N}\left(\psi\left(x,t;q\right)\right)+g(x,t)\end{array}\right\}, \qquad (6)$$

where $q\in\left[0,1\right]$ is an embedding parameter, $\mathcal{H}(q)$ is a nonzero auxiliary function for $q\neq 0$, $\mathcal{H}(0)=0$. Eq (6) is called optimal homotopy equation. Clearly, we have:

$$q=0\Rightarrow\mathcal{H}\left(\psi\left(x,t;0\right),0\right)=\mathcal{L}(\psi(x,t;0))+g(x,t)=0, \qquad (7)$$

$$q=1\Rightarrow\mathcal{H}\left(\psi\left(x,t;1\right),1\right)=\mathcal{H}(1)\left\{\begin{array}{l}\mathcal{L}(\psi(x,t;q))+\\ \mathcal{N}(\psi(x,t;q))+g(x,t)\end{array}\right\}=0. \qquad (8)$$

Clearly, when $q=0$ and $q=1$ it holds that $\psi\left(x,t;0\right)=u_0\left(x,t\right)$ and $\psi\left(x,t;1\right)=u\left(x,t\right)$ respectively. Thus, as q varies from 0 to 1, the solution $\psi\left(x,t;q\right)$ approaches from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is obtained from Eq (3) for $q=0$

$$\mathcal{L}\left(u_0(x,t)\right)+g(x,t)=0,\quad \mathcal{B}\left(u_0,\partial u_0/\partial t\right)=0. \qquad (9)$$

Next, we choose auxiliary function $\mathcal{H}\left(q\right)$ in the form $\mathcal{H}\left(q\right)=qC_1+q^2C_2+\ldots \qquad (10)$

Here $C_1,C_2,\ldots$ are constants to be determined later. To get an approximate solution, we expand $\psi\left(x,t;q,C_i\right)$ in Taylor's series about q in the following manner,

$$\psi\left(x,t;q,C_i\right)=u_0\left(x,t\right)+\sum_{k=1}^{\infty}u_k\left(x,t;C_i\right)q^k, \qquad (11)$$
$$i=1,2,\ldots$$

Substituting Eq. (10) into Eq. (4) and equating the coefficient of like the powers of q, we obtain Zeroth order problem, given by Eq. (6), the first and second order problems are given by Eqs. (11-12) respectively and the general governing equations for $u_k(x,t)$ are given by Eq. (13):

$$\mathcal{L}\left(u_1(x,t)\right)=C_1\mathcal{N}_0\left(u_0(x,t)\right),\qquad \mathcal{B}\left(u_1,\partial u_1/\partial t\right)=0 \qquad (12)$$

$$\mathcal{L}\left(u_2(x,t)\right)-\mathcal{L}\left(u_1(x,t)\right)=C_2\mathcal{N}_0\left(u_0(x,t)\right)+C_1\left[\begin{array}{l}\mathcal{L}\left(u_1(x,t)\right)+\\ \mathcal{N}_1\left(u_0(x,t),u_1(x,t)\right)\end{array}\right], \qquad (13)$$
$$\mathcal{B}\left(u_2,\partial u_2/\partial t\right)=0$$

$$\mathcal{L}\left(u_k(x,t)\right)-\mathcal{L}\left(u_{k-1}(x,t)\right)=C_k\mathcal{N}_0\left(u_0(x,t)\right)+\sum_{i=1}^{k-1}C_i\left[\begin{array}{l}\mathcal{L}\left(u_{k-i}(x,t)\right)\\ +\mathcal{N}_{k-i}\left(\begin{array}{l}u_0(x,t),\\ u_1(x,t),\ldots,\\ u_{k-i}(x,t)\end{array}\right)\end{array}\right], \qquad (14)$$
$$\mathcal{B}\left(u_k,\partial u_k/\partial t\right)=0.\qquad k=2,3,\ldots,$$

where $\mathcal{N}_{k-i}\left(u_0(x,t),u_1(x,t),\ldots,u_{k-i}(x,t)\right)$ is the coefficient of q^{k-i} in the expansion of $\mathcal{N}\left(\psi\left(x,t;q\right)\right)$ about the embedding parameter q.

$$\mathcal{N}\left(\psi\left(x,t;q,C_i\right)\right)=\mathcal{N}_0\left(u_0(x,t)\right)+\sum_{k\geq 1}\mathcal{N}_k\left(u_0,u_1,u_2,\ldots,u_k\right)q^k. \qquad 15)$$

Here u_k for $k\geq 0$ are set of linear equations with the linear boundary conditions, which can be easily solved.

The convergence of the series in Eq. (10) depends upon the auxiliary constants $C_1,C_2,\ldots$. If it is convergent at $q=1$, one has:

$$\tilde{u}\left(x,t;C_i\right)=u_0(x,t)+\sum_{k\geq 1}u_k\left(x,t;C_i\right). \qquad (16)$$

Substituting Eq. (15) into Eq. (1), it results in the following expression for residual:

$$\mathcal{R}\left(x,t;C_i\right)=\mathcal{L}(\tilde{u}(x,t;C_i))+g(x,t)+\mathcal{N}(\tilde{u}(x,t;C_i)). \qquad (17)$$

If $R\left(x,t;C_i\right)=0$, then $\tilde{u}\left(x,t;C_i\right)$ will be the exact solution.

For computing the of auxiliary constants, $C_i, i=1,2,\ldots,m,$ there are many methods like Galerkin's Method, Ritz Method, Least Squares Method and Collocation Method to find the optimal values of $C_i, i=1,2,\ldots,m,$ One can apply the Method of Least Squares as under:

$$\mathcal{J}(C_i)=\int_0^t\int_\Omega \mathcal{R}^2(x,t,C_i)\,dx\,dt, \qquad (18)$$

where R is the residual, $\mathcal{R}\left(x,t;C_i\right)=\mathcal{L}\left(\tilde{u}\left(x,t;C_i\right)\right)+g(x,t)+\mathcal{N}\left(\tilde{u}\left(x,t;C_i\right)\right)$ and

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=...=\frac{\partial J}{\partial C_m}=0. \quad (19)$$

The constants C_i can also be determined by another method as under:

$$\mathcal{R}\left(h_1;C_i\right)=\mathcal{R}\left(h_2;C_i\right)=...=\mathcal{R}\left(h_m;C_i\right)=0, \quad i=1,2,...,m. \quad (20)$$

at any time t, where $h_i \in \Omega$. The convergence, depends upon constants $C_1, C_2,...$, can be optimally identified and minimized by Eq. (18).

3. Application of OHAM for (EWW) Equation.

Consider the (EWW) equation of the form (1)

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}=0, \quad (21)$$

Subject to constant initial condition

$$u(x,0)=3\sec h^2(\frac{x-15}{2}), \quad (22)$$

with exact solution given by

$$u(x,t)=3\sec h^2(\frac{x-15-t}{2}), \quad (23)$$

Zeroth Order Problem

$$\frac{\partial u_0(x,t)}{\partial t}=0,$$

Its solution is

$$u_0(x,t)=3\sec h^2(0.5(x-15). \quad (24)$$

First Order Problem

$$\frac{\partial u_1(x,t)}{\partial t}-(1+C_1)\frac{\partial u_0(x,t)}{\partial t}-C_1 u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}+C_1\frac{\partial^3 u_0(x,t)}{\partial x^2 \partial t}=0. \quad (25)$$

$$u_1(x,0)=0.$$

Its solution is

$$u_1(x,t,\mathrm{C}_1)=-9\mathrm{C}_1\, t\,\mathrm{sech}^4(0.5(x-15)) \tanh(0.5(x-15)). \quad (26)$$

Second Order Problem

$$\frac{\partial u_2(x,t)}{\partial t}-(1+C_1)\frac{\partial u_1(x,t)}{\partial t}-C_2 u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}+C_2\frac{\partial^2 u_0(x,t)}{\partial x^2 \partial t}-C_2\frac{\partial u_0(x,t)}{\partial t}-C_1 u_1(x,t)\frac{\partial u_0(x,t)}{\partial x}-C_1 u_0(x,t)\frac{\partial u_1(x,t)}{\partial x}+C_1\frac{\partial^3 u_1(x,t)}{\partial x^2 \partial t}=0.$$

$$u_2\left(x,0\right)=0 \quad (27)$$

Its solution is

$$\begin{aligned}u_2(x,t,\mathrm{C}_1)=&-6.75tC_1^2\sec h^8(7.5-0.5x)\\&-9C_1 t\mathrm{sech}^4(7.5-0.5x)\tanh(0.5(x-15))\\&-9C_1^2 t\mathrm{sech}^4(7.5-0.5x)\tanh(0.5(x-15))\\&-9C_2 t\mathrm{sech}^4(7.5-0.5x)\tanh(0.5(x-15))\\&-31.5\,C_1^2 t\mathrm{sech}^6(7.5-0.5x)\tanh^2(0.5(x-15))\\&+36\,C_1^2 t\mathrm{sech}^4(7.5-0.5x)\tanh^2(0.5(x-15)).\end{aligned} \quad (28)$$

Third Order Problem

$$\frac{\partial u_3(x,t)}{\partial t}-(1+C_1)\frac{\partial u_2(x,t)}{\partial t}-C_3 u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}+C_3\frac{\partial^3 u_0(x,t)}{\partial x^2 \partial t}-C_3\frac{\partial u_0(x,t)}{\partial t}-C_2 u_1(x,t)\frac{\partial u_0(x,t)}{\partial x}-C_2 u_0(x,t)\frac{\partial u_1(x,t)}{\partial x}+C_2\frac{\partial^3 u_1(x,t)}{\partial x^2 \partial t}-C_2\frac{\partial u_1(x,t)}{\partial t}-C_1 u_2(x,t)\frac{\partial u_0(x,t)}{\partial x}-C_1 u_1(x,t)\frac{\partial u_1(x,t)}{\partial x}-C_1 u_0(x,t)\frac{\partial u_2(x,t)}{\partial x}+C_1\frac{\partial^3 u_2(x,t)}{\partial x^2 \partial t}=0.$$

$$u_3\left(x,0\right)=0 \quad (29)$$

Its solution $u_3(x,t,\mathrm{C}_1,\mathrm{C}_2,\mathrm{C}_3)$ is obtained in the same manner. The third order approximate solution is of the form.

$$\tilde{u}\left(x,t,C_1,C_2\right)=u_0\left(x,t\right)+u_1\left(x,t,C_1\right)+u_2\left(x,t,C_1,C_2\right)+u_3\left(x,t,C_1,C_2,C_3\right) \quad (30)$$

For the calculations of the constants C_1, C_2 and C_3 using the Method Least Squares we have computed that

$$\mathrm{C}_1=-2.6832331347988335\times10^{-8},\quad \mathrm{C}_2=3917.7906025323711, \text{ and } \quad \mathrm{C}_3=-32847.8147637734. \quad (31)$$

The 3rd order OHAM solution yields very encouraging results after comparing with 3rd order NIM andADM solution [18].

Table 1.1: Comparison between the absolute error of the solution of EWW equation by He's variational iteration method (VIM) and Adomian decomposition methods (ADM) and optimal homotopy asymptotic method (OHAM) at various values of t and = 0.

t	VIM	ADM	OHAM
0.01	3.668×10^{-9}	3.668×10^{-9}	3.33195×10^{-9}
0.02	7.333×10^{-9}	7.334×10^{-9}	6.66023×10^{-9}
0.03	1.099×10^{-8}	1.099×10^{-8}	9.98485×10^{-9}
0.04	1.465×10^{-8}	1.465×10^{-8}	1.33058×10^{-8}
0.05	1.830×10^{-8}	1.830×10^{-8}	1.66231×10^{-8}
0.1	3.652×10^{-8}	3.652×10^{-8}	3.31549×10^{-8}

Table 1.2:Comparison between the absolute error of the solution of EWW equation by He's variational iteration method (VIM) and Adomian decomposition methods (ADM) and optimal homotopy asymptotic method (OHAM) at various values of t and $x = 5$

t	VIM	ADM	OHAM
0.01	5.382×10^{-7}	5.429×10^{-7}	5.32925×10^{-7}
0.02	1.075×10^{-6}	1.085×10^{-6}	1.06531×10^{-6}
0.03	1.612×10^{-6}	1.627×10^{-6}	1.67319×10^{-6}
0.04	2.149×10^{-6}	2.168×10^{-6}	2.12844×10^{-6}
0.05	2.685×10^{-6}	2.709×10^{-6}	2.92244×10^{-6}
0.1	5.357×10^{-6}	5.405×10^{-6}	5.83132×10^{-6}

Table 1.3: Comparison between the absolute error of the solution of EWW equation by He's variational iteration method (VIM) and Adomian decomposition methods (ADM) and optimal homotopy asymptotic method (OHAM) at various values of t and $x = 10$

t	NIM	ADM	OHAM
0.01	3.611×10^{-5}	4.852×10^{-5}	2.21357×10^{-5}
0.02	7.228×10^{-5}	9.698×10^{-5}	4.43255×10^{-5}
0.03	1.085×10^{-4}	1.453×10^{-4}	6.65695×10^{-5}
0.04	1.448×10^{-4}	1.936×10^{-4}	8.88676×10^{-5}
0.05	1.811×10^{-4}	2.418×10^{-4}	1.1122×10^{-4}
0.1	3.637×10^{-4}	4.819×10^{-4}	2.23787×10^{-4}

Table 1.4: Comparison between the absolute error of the solution of EWW equation by He's variational iteration method (VIM) and Adomian decomposition methods (ADM) and optimal homotopy asymptotic method (OHAM) at various values of t and $x = 15$

t	NIM	ADM	OHAM
0.01	5.137×10^{-5}	6.000×10^{-6}	3.03829×10^{-7}
0.02	2.055×10^{-5}	2.400×10^{-5}	1.21531×10^{-6}
0.03	4.623×10^{-5}	5.400×10^{-5}	2.73445×10^{-6}
0.04	8.220×10^{-5}	9.600×10^{-5}	4.86123×10^{-6}
0.05	1.284×10^{-4}	1.500×10^{-4}	7.59564×10^{-6}
0.1	5.137×10^{-4}	6.000×10^{-4}	3.03816×10^{-5}

Table 1.5:Comparison between the absolute error of the solution of EWW equation by He's variational iteration method (VIM) and Adomian decomposition methods (ADM) and optimal homotopy asymptotic method (OHAM) at various values of t and $x = 20$

t	NIM	ADM	OHAM
0.01	3.605×10^{-5}	4.860×10^{-5}	2.05582×10^{-5}
0.02	7.205×10^{-5}	9.728×10^{-5}	4.11931×10^{-5}
0.03	1.080×10^{-4}	1.460×10^{-4}	6.19047×10^{-5}
0.04	1.438×10^{-4}	1.948×10^{-4}	8.26931×10^{-5}
0.05	1.797×10^{-4}	2.437×10^{-4}	1.03558×10^{-4}
0.1	3.580×10^{-4}	4.894×10^{-4}	2.09041×10^{-4}

Table 1.6: Comparison between the absolute error of the solution of EWW equation by He's variational iteration method (VIM) and Adomian decomposition methods (ADM) and optimal homotopy asymptotic method (OHAM) at various values of t and $x = 15$

t	NIM	ADM	OHAM
0.01	5.387×10^{-7}	5.434×10^{-7}	5.42223×10^{-7}
0.02	1.078×10^{-6}	1.087×10^{-6}	1.08499×10^{-6}
0.03	1.617×10^{-6}	1.632×10^{-6}	1.6283×10^{-6}
0.04	2.158×10^{-6}	2.177×10^{-6}	2.17216×10^{-6}
0.05	2.699×10^{-6}	2.722×10^{-6}	2.71657×10^{-6}
0.1	5.412×10^{-6}	5.459×10^{-6}	5.44683×10^{-6}

4. Application of OHAM for Burger Equation.

Let us consider Burger Equation of the form (2).

$$\frac{\partial u(x,t)}{\partial t} + \alpha u^{\delta}(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (32)$$

With initial condition given by

$$u(x,0) = 0.5 + 0.5\tanh\left(\frac{-\alpha\,\delta}{2(\delta+1)}x\right) \qquad (33)$$

Case 1: when $\alpha = 1$ and $\delta = 1$

For $\alpha = 1$ and $\delta = 1$ the above equation takes the form

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (34)$$

Subject to constant initial condition

$$u(x,0) = 0.5 - 0.5\tanh(0.25x) \qquad (35)$$

The exact solution of equation (24) with given condition is given by

$$u(x,t) = 0.5 - 0.5\tanh[0.25(x-0.5t)]\ , \qquad (36)$$

following the basic idea of OHAM presented in preceding Section we start with

$$L(\phi(x,t,q)) = \frac{\partial \phi(x,t,q)}{\partial t}, \qquad (37)$$

$$N(\phi(x,t,q)) = u(x,t)\frac{\partial u(x,t,q)}{\partial x} - \frac{\partial^2 u(x,t,q)}{\partial x^2}. \qquad (38)$$

The initial condition is:

$$\phi(x,0,q) = 0.5 - 0.5\tanh[0.25(x-0.5t)]. \qquad (39)$$

Zeroth Order Problem

$$\frac{\partial u_0(x,t)}{\partial t} = 0\,,$$

Its solution is

$$u_0(x,0) = 0.5 - 0.5\tanh(0.25x). \qquad (40)$$

First Order Problem

$$\frac{\partial u_1(x,t)}{\partial t}-(1+C_1)\frac{\partial u_0(x,t)}{\partial t}-$$
$$C_1u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}+C_1\frac{\partial^2 u_0(x,t)}{\partial x^2}=0$$
$$u_1(x,0)=0. \tag{41}$$

Its solution is

$$u_1(x,t,C_1)=-t(0.0625C_1\operatorname{sech}^2(0.25x)). \tag{42}$$

Second Order Problem

$$\frac{\partial u_2(x,t)}{\partial t}-(1+C_1)\frac{\partial u_1(x,t)}{\partial t}-C_2u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}$$
$$+C_2\frac{\partial^2 u_0(x,t)}{\partial x^2}-C_2\frac{\partial u_0(x,t)}{\partial t}-C_1u_1(x,t)\frac{\partial u_0(x,t)}{\partial x}$$
$$-C_1u_0(x,t)\frac{\partial u_1(x,t)}{\partial x}+C_1\frac{\partial^2 u_1(x,t)}{\partial x^2}=0.$$
$$u_2(x,0)=0 \tag{43}$$

Its solution is

$$\begin{aligned}u_2(x,t,C_1)=&((-0.0625C_1(1+C_1)-0.0625C_2)\\&t\sec h^2(0.25x)+\sec h^5(0.25x)\\&(0.00195313C_1^{\ 2}t^2\sinh(0.75x))+\\&\sec h^4(0.25x)(0.00195313\\&C_1^{\ 2}t^2\tanh(0.75x))).\end{aligned} \tag{44}$$

Third Order Problem

$$\frac{\partial u_3(x,t)}{\partial t}-(1+C_1)\frac{\partial u_2(x,t)}{\partial t}-$$
$$C_3u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}+C_3\frac{\partial^2 u_0(x,t)}{\partial x^2}-$$
$$C_3\frac{\partial u_0(x,t)}{\partial t}-C_2u_1(x,t)\frac{\partial u_0(x,t)}{\partial x}-$$
$$C_2u_0(x,t)\frac{\partial u_1(x,t)}{\partial x}+C_2\frac{\partial^2 u_1(x,t)}{\partial x^2}-$$
$$C_2\frac{\partial u_1(x,t)}{\partial t}-C_1u_2(x,t)\frac{\partial u_0(x,t)}{\partial x}-$$
$$C_1u_1(x,t)\frac{\partial u_1(x,t)}{\partial x}+C_1\frac{\partial^2 u_0(x,t)}{\partial x^2}=0.$$
$$u_3(x,0)=0 \tag{45}$$

Its solution is

$$\begin{aligned}u_3(x,t,C_1,C_2,C_3)=&(\frac{1}{(1+e^{0.5x})^2}t\sec h^2(0.25x)\\&(-0.0625C_2-0.0625C_3+C_1(-0.0625+C_1(-0.125-\\&0.015625t)+C_2(-0.125-0.015625t)-0.000651042C_1^{\ 2}\\&(5.0718+t)(18.9282+t))+((288C_2+288C_3+\\&C_1(288+576C_2+C_1(576-24t)-24C_2t-\\&3C_1^{\ 2}(-6.58301+t)(14.583+t)\cosh(0.25x)+\\&(-96C_2-96C_3+C_1(-96-192C_2+C_1(-192-24t)\\&-24C_2t+5C_1^{\ 2}(-7.396+t)(2.596+t)))\sinh(0.25x))\\&-24C_2t+5C_1^{\ 2}(-7.396+t)(2.596+t)))\sinh(0.25x)\\&(-0.000651042\cosh(0.75x)-0.000651042\sinh(0.75x)))).\end{aligned} \tag{46}$$

Adding equations (40, 42, 44, 46) we obtain:

$$\begin{aligned}\tilde{u}(x,t,C_1,C_2)=&u_0(x,t)+u_1(x,t,C_1)+\\&u_2(x,t,C_1,C_2)+u_3(x,t,C_1,C_2,C_3)\end{aligned} \tag{47}$$

For the calculations of the constants C_1, C_2 and C_3 using the Method Least Squares we have computed that

$C_1=-1.0000840\ \ 70827354,$

$C_2=9.46571385\ \ 1467995\ \times 10^{-8}$, *and*

$C_3=-1.98940934\ \ 674601\ \times 10^{-11}$.

Putting the values of these constants into equation (34) the third order approximate solution using of OHAM is

$$\begin{aligned}u_3(x,t)=&(0.5-5.26078452\ 47854\times 10^{-6}\\&t\operatorname{sech}^2(0.25x)+\frac{1}{(1+e^{0.5x})^2}\\&(6.3600904\ 15451543\times 10^{-9}+\\&(1.3153067\ 008227217\times 10^{-6}\\&+0.00065120\ 5881306229\ 2t)t+\\&(1.2720180\ 853076355\times 10^{-8}-\\&0.00260482\ 3525224917\ t^2)\cosh(0.5x)+\\&(6.3600904\ 2653818\times 10^{-9}+\\&(-1.315306\ 7008227217\times 10^{-6}+\\&0.00065120\ 5881306229\ 2t)t)\cosh(x)\\&+(1.2720180\ 853076355\times 10^{-8}-\\&0.00260482\ 3525224917\ t^2)\sinh(0.5x)\\&+(6.3600904\ 2653818\times 10^{-9}+\\&(-1.315306\ 7008227217\times 10^{-9}+\\&0.00065120\ 5881306229\ 2t)t)\sinh(x)-\\&0.5\tanh(0.25x)+0.00781381\ 3661895406\ t^2\\&\operatorname{sech}^2(0.25x)\tanh(0.25x)+\operatorname{sech}^2(0.25x)\\&(0.0625052\ 5442670962\ t)).\end{aligned} \tag{38}$$

The 3rd order OHAM solution yields very encouraging results after comparing with 4th order ADM solution [16]

Table 2.1: Comparison of absolute errors of 3rd order OHAM solution and 4th order ADM solution for Burger equation for $x = 0.1$ and $t \in [0, 2]$

t	Exact solution	Absolute Error ADM	Absolute Error OHAM
0.5	0.518741	6.34216×10^{-8}	5.32631×10^{-8}
1.0	0.549834	2.02886×10^{-6}	7.98928×10^{-8}
2.0	0.610639	6.42801×10^{-5}	3.2441×10^{-5}

Table 2: Comparison of absolute errors of 3rd order OHAM solution and 4th order ADM solution for Burger equation for $x = 0.5$ and $t \in [0, 2]$

t	Exact solution	Absolute Error ADM	Absolute Error OHAM
0.5	0.468791	5.66705×10^{-8}	5.82744×10^{-8}
1.0	0.50000	1.8471×10^{-6}	3.89112×10^{-6}
2.0	0.562177	6.06928×10^{-5}	7.40943×10^{-5}

Table 2.3 Absolute errors of 3rd order OHAM solution for various values of x and $t = 0.003$ *and* $t = 0.1$

x	Absolute Error for $t = 0.003$	Absolute Error for $t = 0.1$
-4	7.88258×10^{-15}	3.00377×10^{-10}
-2	8.54872×10^{-15}	2.023×10^{-9}
0	4.55191×10^{-15}	1.02224×10^{-10}
2	1.11022×10^{-15}	1.98526×10^{-9}
4	9.57567×10^{-16}	3.6574×10^{-10}

Table 2.4 Absolute errors of 3rd order OHAM solution for various values of x and $t = 0.5$ *and* $t = 0.1$

x	Absolute Error for $t = 0.5$	Absolute Error for $t = 1$
-4	1.86591×10^{-7}	2.68667×10^{-6}
-2	1.2447×10^{-6}	1.96202×10^{-5}
0	7.37402×10^{-8}	2.10382×10^{-6}
2	1.26475×10^{-6}	2.0391×10^{-5}
4	2.37125×10^{-7}	4.14971×10^{-6}

Case 2: when $\alpha = 1$ *and* $\delta = 2$

For $\alpha = 1$ and $\delta = 22$ equation (2) takes the form

$$\frac{\partial u(x,t)}{\partial t} + u^2(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (34)$$

Subject to constant initial condition

$$u(x,0) = 0.5 - 0.5\tanh(\frac{x}{3}) \qquad (35)$$

The exact solution of equation (24) with given condition is given by

$$u(x,t) = 0.5 - 0.5\tanh[\frac{1}{4}(x - \frac{1}{3}t)] \ , \qquad (36)$$

Using same lines as above the third order approximate solution using OHAM is obtained and absolute errors for various values of $x$and t are given in table (2.5-2.6).

Table 2.5: Absolute error of the solution of Burger equation optimal homotopy asymptotic method (OHAM) at $x = 0.1$ and various values of t

t	Exact solution	OHAM solution	Absolute error
0.1	0.699207	0.699207	1.02602×10^{-11}
0.2	0.703168	0.703168	3.52315×10^{-9}
0.3	0.707107	0.707107	2.39034×10^{-8}
0.4	0.711024	0.711024	8.5422×10^{-8}
0.5	0.714919	0.714919	2.23819×10^{-7}
0.6	0.718791	0.718791	4.86729×10^{-7}
0.7	0.722639	0.722638	9.34107×10^{-7}
0.8	0.726464	0.726462	1.63865×10^{-6}
0.9	0.730263	0.73026	2.68617×10^{-6}
1.0	0.734037	0.734033	4.17601×10^{-6}

Table 2.6: Absolute error of the solution of Burger equation optimal homotopy asymptotic method (OHAM) at $x = 0.5$ and various values of t

t	Exact solution	OHAM solution	Absolute error
0.1	0.650264	0.650264	3.16105×10^{-10}
0.2	0.654428	0.654428	1.15677×10^{-9}
0.3	0.658578	0.658578	7.62659×10^{-10}
0.4	0.662715	0.662715	1.33499×10^{-8}
0.5	0.666837	0.666837	4.97044×10^{-8}
0.6	0.670944	0.670944	1.28683×10^{-7}
0.7	0.675035	0.675035	2.7546×10^{-7}
0.8	0.679109	0.679109	5.22082×10^{-7}
0.9	0.683166	0.683165	9.08017×10^{-7}
1.0	0.687205	0.687204	1.48069×10^{-6}

Conclusion

In this paper, the OHAM has been successfully implemented for the approximate solution of Burger and EWW Equations. For EWW equation the results obtained by OHAM are very consistent in comparison with ADM and VIM. For Burger equationsthe third order approximate solutions results of proposed method are very encouraging and agrees to the fourth order approximate solution by ADM.

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11/10/2012