Numerical solution of linear Fredholm integral equations

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Abstract: In this paper, numerical solution of linear Fredholm integral equations of the second kind is considered by two methods. The methods are developed by means of the Sinc-collocation method and shifted Chebyshev polynomial method. Some numerical examples are presented to illustrate the method.

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1. Introduction

Many initial and boundary value problems can be transformed into integral equations and in many cases, we cannot solve this equations analytically to find an exact solution. So that by using numerical methods we try to find the approximate solution of these equations. Several authors have considered the numerical solution of the integral equations with different methods ([1 ,2,5,7,10,11]).

This paper consists of two parts. In part I, we study the numerical solution of system of linear Fredholm integral equations of the second kind by means of Sinc-collocation method, this method consists of reducing the system of Fredholm integral equations to a set of algebraic equations with unknown coefficients by using the properties of Sinc function. In part II, we study the numerical solution of linear Fredholm integral equations by shifted Chebyshev polynomial method which transforms Fredholm integral equation into a matrixequation.

Part I: Numerical solution of system of linear Fredholm integral equations by Sinc- collocation method

This part consists of three sections. Section 1, outlines some of the main properties of Sinc function which are necessary for the formulation of the problem. In section 2, we illustrate how Sinc-collocationmethod may be used to replace system of linear Fredholm integral equations into system of linear algebraic equations. Finally in section 3, we will illustrate the method by some numerical examples.

Now, we consider the system of linear Fredholm integral equations of the form:

$$\Phi(x)=F(x)+\int_{\Gamma}H(x,t)\Phi(t)\,dt\,,\quad x\in\Gamma=[a,b] \tag{1.1}$$

Where

$\Phi(x)=[\phi_1(x),\phi_2(x),\ldots,\phi_n(x)]^T$, $F(x)=[f_1(x),f_2(x),\ldots,f_n(x)]^T$ and

$$H(x,t)=[H_{ij}(x,t)]\quad ,i,j=1,2,\ldots,n,$$

is the unknown function $\Phi(x)$ F(x) and H(x, t) are known functions and to be determined.

1. Sinc function and its properties [3]

The Sinc function is defined on the whole real line by:

$$Sinc(z)=\begin{cases}\frac{\sin(\pi z)}{\pi z}, & z\neq 0,\\ 1, & z=0.\end{cases} \tag{1.2}$$

Now, for h > 0 and integer j , we define the jthSinc function with step size h by:

$$S(j,h)(z)=\frac{\sin(\pi(z-jh)/h)}{\pi(z-jh)/h},\qquad j=0,\pm1,\pm2,\ldots. \tag{1.3}$$

is given by $z_k=kh$ The Sinc function form for the interpolation points

$$S(j,h)(kh)=\delta^{(0)}_{jh}=\begin{cases}1, & k=j,\\ 0, & k\neq j.\end{cases} \tag{1.4}$$

Is defined on the real line, then for h > 0 the series ϕ If

$$C(\phi,h)(z)=\sum_{j=-\infty}^{\infty}\phi(jh)\frac{\sin(\pi(z-jh)/h)}{\pi(z-jh)/h}. \tag{1.5}$$

, whenever this series ϕ is called the Whittaker cardinal expansion of

is approximated by using the finite number of terms in ϕ converges ,

(1.5). For positive integer N, we define

$$C(\phi,h)(z)=\sum_{j=-N}^{N}\phi(jh)\frac{\sin(\pi(z-jh)/h)}{\pi(z-jh)/h}. \quad (1.6)$$

Γ. Now, to construct approximation on the interval

Let

$$\psi(z)=\omega=\ln(\frac{z-a}{b-z}), \quad (1.7)$$

be a conformal mapping which maps the simply connected domain D onto a

where D_d strip region

$$D=\{z=x+iy:\left|\arg(\frac{z-a}{b-z})\right|<d\le\frac{\pi}{2}\}, \quad (1.8)$$

$$D_d=\{\omega=\alpha+i\beta:|\beta|<d\le\frac{\pi}{2}\}, \quad (1.9)$$

such that

$\psi((a,b))=(-\infty,\infty)$, $\lim_{z\to a}\psi(z)=-\infty$ and $\lim_{z\to b}\psi(z)=\infty$.

are Γ for $z\in D$ For the Sinc method, the basis functions on the interval

from the composite translated Sinc functions

$$S_j(z)=S(j,h)o\psi(z)=\frac{\sin(\pi(\psi(z)-jh)/h)}{\pi(\psi(z)-jh)/h}. \quad (1.10)$$

The function

$$z=\psi^{-1}(\omega)=\frac{a+be^{\omega}}{1+e^{\omega}}, \quad (1.11)$$

on the real ψ^{-1} We define the range of $\omega=\psi(z)$. is an inverse mapping of

line as:

$$\Gamma=\{\eta(u)=\psi^{-1}(u)\in D: \quad -\infty<u<\infty\}. \quad (1.12)$$

because they x_k will be denoted by $z_k\in\Gamma$ in D

The Sinc- collocation points

on the real line, the image $\{kh\}_{k=-\infty}^{\infty}$ are real. For the evenly spaced nodes

which corresponds to these nodes is denoted by:

$$x_k=\psi^{-1}(kh)=\frac{a+be^{kh}}{1+e^{kh}}, \quad k=\pm1,\pm2,\ldots. \quad (1.13)$$

Now, we consider the main definition and theorem.be the set of all analytic functions u in D, for which $L_\alpha(D)$ **Definition 1.** Let

there exists a constant C such that

$$|u(z)|\le C\frac{|\rho(z)|^{\alpha}}{[1+|\rho(z)|]^{2\alpha}}, \qquad z\in D, \quad 0<\alpha\le1, \quad (1.14)$$

be a positive integer and h be selected by the $\frac{u}{\psi'}\in L_\alpha(D)$, N **Theorem 1.** Let

formula

$$h=(\frac{2\pi d}{\alpha N})^{1/2}. \quad (1.15)$$

independent on N, such that C_1 Then there exists positive constant

$$\left|\int_\Gamma u(z)\,dz-h\sum_{k=-N}^{N}\frac{u(z_k)}{\psi'(z_k)}\right|\le C_1\,e^{(-2\pi\alpha dN)^{1/2}}. \quad (1.16)$$

2. The approximate solution of system of Fredholm integral equations

We consider the ith equation of (1.1):

$$\phi_i(x)=f_i(x)+\sum_{j=1}^{n}\int_\Gamma H_{ij}(x,t)\phi_j(t)\,dt, \qquad i=1,2,\ldots,n. \quad (1.17)$$

For the second term on the right-hand side of (1.17), we suppose that

then by using Theorem 1, we obtain

$$\frac{H_{ij}(x,.)}{\psi'}\in L_\alpha(D),$$

$$\int_\Gamma H_{ij}(x,t)\phi_j(t)\,dt\approx h\sum_{\ell=-N}^{N}\frac{H_{ij}(x,t_\ell)}{\psi'(t_\ell)}\phi_{j\ell}, \quad (1.18)$$

where

$\phi_j(x_\ell)$ denotes an approximate value of $\phi_{j\ell}$

Using (1.17) and (1.18) we obtain

$$\phi_i(x)-h\sum_{j=1}^{n}[\sum_{\ell=-N}^{N}\frac{H_{ij}(x,t_\ell)}{\psi'(t_\ell)}\phi_{j\ell}]\approx f_i(x), \qquad i=1,2,\ldots,n. \quad (1.19)$$

$\phi_{j\ell}$, $\ell=-N,-N+1,\ldots,N;\ j=1,2,\ldots,n,$ There are n×(2N +1) unknowns

to be determined in (1.19). In order to determine these n × (2N + 1) unknowns, we apply the collocation method. Thus by setting

areSinc-collocation points: x_k in (1.19) where $x_k, k=-N,\ldots,N$

$\rho(z)=e^{\psi(z)}$ .where

$$x_k = \eta(kh) = \psi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}. \tag{1.20}$$

From (1.19) and (1.20) we obtain the following system of n × (2N + 1)

linear equations with n × (2N + 1) unknowns $\phi_{j\ell}$, $\ell = -N, -N+1, \dots, N;\ j = 1, 2, \dots, n,$

$$\phi_{ik} - h\sum_{j=1}^{n}[\sum_{\ell=-N}^{N} \frac{H_{ij}(x_k, t_\ell)}{\psi'(t_\ell)}\phi_{j\ell}] = f_i(x_k), \quad i = 1,2,\dots,n\,; k = -N,\dots,N. \tag{1.21}$$

and $\widetilde{H}_{ij} = [\frac{H_{ij}(x_k, t_\ell)}{\psi'(t_\ell)}]$ We denote

$$B_{ij} = \begin{cases} I - h\widetilde{H}_{ij}, & i = j, \\ -h\widetilde{H}_{ij}, & i \neq j, \end{cases}$$

which are the square matrices of order (2N + 1) × (2N + 1), then the system of can be expressed in a $\phi_{j\ell}$ linear equations (1.21) unknown coefficients

matrix form

$$B\widetilde{\Phi} = P, \tag{1.22}$$

where

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix},$$

$P = [f_1(x_{-N}), \dots, f_1(x_N), \dots, f_n(x_{-N}), \dots, f_n(x_N))]^T$, $\widetilde{\Phi} = [\phi_{1\ell}, \dots, \phi_{n\ell}]^T$, $\ell = -N, \dots, N$

$\phi_{j\ell}$

By solving the linear system (1.22), we obtain an approximate solution

of the system of integral equations $\phi_j(x_\ell)$ corresponding to the exact solution

(1.1) at the Sinc points.

3. Numerical examples

In this section we will illustrate the above results by some examples. The examples have been solved by presented method with different values of N.

The errors

$\alpha = 1$ *and*, $d = \pi/2$, *which yields* $h = \dfrac{\pi}{\sqrt{N}}$.

In all examples we take

are reported on the set of Sinc grid points

$$S = \{x_{-N}, \dots, x_0, \dots, x_N\}\ , \quad x_k = \frac{a+be^{kh}}{1+e^{kh}}\ , \qquad k = -N, \dots, N.$$

The maximum error on the Sinc grid points is

$$\left\| E_\Phi^s(h) \right\|_\infty = \max_{-N \le j \le N} \left| \Phi(x_j) - \Phi_N(x_j) \right|. \tag{1.23}$$

Example 1.

Consider the following system of Fredholm integral equations

$$\phi_1(x) = f_1(x) + \int_0^1 (x-t)^3 \phi_1(t)\,dt + \int_0^1 (x-t)^2 \phi_2(t)\,dt,$$

$$\phi_2(x) = f_2(x) + \int_0^1 (x-t)^4 \phi_1(t)\,dt + \int_0^1 (x-t)^3 \phi_2(t)\,dt, \tag{1.24}$$

with

$$f_1(x) = \frac{1}{20} - \frac{11}{30}x + \frac{5}{3}x^2 - \frac{1}{3}x^3, \quad f_2(x) = -\frac{1}{30} - \frac{41}{60}x + \frac{3}{20}x^2 + \frac{23}{12}x^3 - \frac{1}{3}x^4$$

$\phi_1(x) = x^2$, $\phi_2(x) = -x + x^2 + x^3$. and the exact solution

We solved Example 1 for different values of N and the maximum of absolute errors on the Sinc grid Sare tabulated in Table 1. This table indicates that as N increases the errors are decreasing more rapidly where excellent results are shown.

Example 2.

Consider the following system of Fredholm integral equations

$$\phi_1(x) = \frac{11}{6}x + \frac{11}{15} - \int_0^1 (x+t)\phi_1(t)\,dt - \int_0^1 (x+2t^2)\phi_2(t)\,dt,$$

$$\phi_2(x) = \frac{5}{4}x^2 + \frac{1}{4}x - \int_0^1 xt^2\phi_1(t)\,dt - \int_0^1 x^2 t\,\phi_2(t)\,dt, \tag{1.25}$$

$\phi_1(x) = x$ *and* $\phi_2(x) = x^2$. with exact solution

The approximate solution is calculated for different values of N and the maximum of absolute errors on the Sinc grid S are tabulated in Table 2.

Part II: Numerical solution of linear Fredholm integral equationsby shifted Chebyshev polynomial method

This part consists of two sections. In section 1, we present shifted Chebyshev polynomial method. Section2, is devoted to introduce the numerical solution of three examples by using shifted Chebyshev polynomial method and Sinc- collocation method.

Consider the following linear Fredholm integral equation:

$$\varphi(x) = f(x) + \lambda \int_0^1 K(x,t)\varphi(t)\,dt, \quad x,t \in [0,1], \tag{2.1}$$

$\varphi(x)$ is a real parameter and λ where f(x), K(x, t) are given functions,
is unknown function.

1. Shifted Chebyshev polynomial method

In this section we will study the approximate solution of equation (2.1) by means of shifted Chebyshev polynomial method.
of equation (2.1) can be represented by truncated $\varphi(x)$ The unknown function
Chebyshev series as follows:

$$\varphi(x) = \sum_{j=0}^{N}{}' \, a_j T_j^*(x), \qquad 0 \le x \le 1 \tag{2.2}$$

a_j denoted the shifted Chebyshev polynomial of the first kind, $T_j^*(x)$ where
is a sum whose first term is Σ' are the unknown Chebyshev coefficients,
halved and N is any positive integer.

of equation (2.1) and K(x, t) can be expressed $\varphi(x)$ Suppose that the solution
as a truncated Chebyshev series. Then (2.2) can be written in the following form:

$$\varphi(x) = T^*(x)\,A, \tag{2.3}$$

where

$T^*(x) = [T_0^*(x) \quad T^*{}_1(x) \quad \dots \quad T_N^*(x)]$, $A = [\frac{a_0}{2} \quad a_1 \quad \dots \quad a_N]^T$,

Now, K(x, t) can be expanded by chebyshev series as follows:

$$K(x_i,t) = \sum_{r=0}^{N}{}'' \, k_r(x_j) T_r^*(t),$$

are the x_i denotes a sum with first and last terms halved Σ'' where
thechebyshev collocation points defined by

$$x_i = \frac{1}{2}[1+\cos(\frac{i\pi}{N})], \qquad i = 0,1,\dots,N, \tag{2.4}$$

are determined by the following relation $k_r(x_j)$ and Chebyshev coefficients
[4,9]:

$$k_r(x_j) = \frac{2}{N}\sum_{j=0}^{N}{}'' \, K(x_i,t_j)T_r^*(t_j), \quad t_j = \frac{1}{2}[1+\cos(\frac{j\pi}{N})].$$

given by $K(x_i,t)$ Then the matrix representation of

$$K(x_i,t) = K(x_i)\,T^*(t)^T, \tag{2.5}$$

where

$$K(x_i) = [\frac{k_0(x_i)}{2} \quad k_0(x_i) \quad \dots \quad k_{N-1}(x_i) \quad \frac{k_N(x_i)}{2}].$$

By substituting from Chebyshev collocation points defined by (2.4) into equation (2.1), we obtain a matrix equation of the form

$$\Phi = F + \lambda I, \tag{2.6}$$

where I(x) denotes the integral part of equation (2.1) and

$$\Phi = \begin{pmatrix} \varphi(x_0) \\ \varphi(x_1) \\ \vdots \\ \varphi(x_N) \end{pmatrix}, \quad F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}, \quad I = \begin{pmatrix} I(x_0) \\ I(x_1) \\ \vdots \\ I(x_N) \end{pmatrix}.$$

When we substitute from Chebyshev collocation

$$\Phi = T^* A. \tag{2.7}$$

points (2.4) into (2.3), the becomes Φ matrix

for i = 0, 1, ...,N , j = 0, 1, ...,N and $I(x_i)$ Substituting from (2.3) and (2.5) in using the following relation [9],

$$Z = \int_0^1 T^*(t)^T \, T^*(t)\, dt = [\int_0^1 T_i^*(t) T_j^*(t)\, dt] = \frac{1}{2}[z_{ij}],$$

where

$$z_{ij} = \begin{cases} \frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}, & \text{for even } i+j, \\ 0, & \text{for even } i+j, \end{cases}$$

we obtain

$$I(x_i) = K(x_i) Z A. \tag{2.8}$$

Therefore, we obtain the matrix I in terms of Chebyshev coefficients matrix in the following form:

$$I = K Z A, \tag{2.9}$$

where

$$K = [k(x_0) \quad k(x_0) \quad \dots \quad k(x_N)\ ]^T.$$

Now, by using the relation (2.7) and (2.9), the integral equation (2.1) transforms into a matrix equation which is given by:

$$T^* A - \lambda KZA = F. \tag{2.10}$$

The matrix equation (2.10) corresponds to a system of (N + 1) linear algebraic equations with (N + 1) unknown Chebyshev coefficients. Thus the unknown can be computed, hence we obtain the approximate solution. a_j coefficients

Particularly : If we apply Sinc-collocation method which is given in part I in case of linear Fredholm integral equation (2.1) we obtain the following system $\varphi_{1\ell}$, $\ell = -N, \dots, N$: of (2N +1) linear equations with (2N +1) unknowns

$$\phi_{1k} - h \sum_{\ell=-N}^{N} \frac{K_{11}(x_k, t_\ell)}{\psi'(t_\ell)} \varphi_{1\ell} = f_1(x_k), \quad k = -N, \dots, N. \tag{2.11}$$

2. Numerical examples

In this section we present three examples to illustrate the above results.

Example 1.

Consider the following linear Fredholm integral equation of the second kind

$$\varphi(x) = x^2 - \frac{4}{3}x + \frac{1}{4} + \int_0^1 (x+t)\varphi(t)\, dt, \qquad 0 \le x \le 1, \tag{2.12}$$

$\varphi(x) = x^2 + 1$. with the exact solution

The numerical solution of equation (2.12) in case of shifted Chebyshev polynomial method and Sinc-collocation method is given in Tables 3 and 4.

Example 2.

Consider the following linear Fredholm integral equation with exact solution

$$\varphi(x) = x(x-1)$$

$$\varphi(x) = \frac{1}{4} - x + \int_0^1 (3t - 6x^2)\varphi(t)\, dt, \qquad 0 \le x \le 1. \tag{2.13}$$

The numerical solution of equation (2.13) in case of shifted Chebyshev polynomial method and Sinc-collocation method is given in Tables 5 and 6.

Example 3.

Consider the following linear Fredholm integral equation with exact solution

$$\varphi(x) = e^x(2x - 2/3)$$

$$\varphi(x) = 2xe^x - 2\int_0^1 e^{(x-t)}\, \varphi(t)\, dt, \qquad 0 \le x \le 1. \tag{2.14}$$

The numerical solution of equation (2.14) in case of shifted Chebyshev polynomial method and Sinc – collocation method is given in Tables 7 and 8.

Table 1. Numerical results of Example 1 in part I

N	h	$\left\| E_{\phi_1}^S(h) \right\|_\infty$	$\left\| E_{\phi_2}^S(h) \right\|_\infty$
5	1.404963	3.150485 E−3	1.346648 E−3
10	0.9934589	1.827180 E−4	2.301931E−4
15	0.8111557	3.388524 E−5	7.402897 E−5
20	0.7024815	6.556511 E−6	4.351139 E−6
25	0.6283185	2.384186 E−7	7.152557 E−7
30	0.5735737	7.078052 E−8	1.192093 E−8
35	0.5310261	4.103521 E−9	7.326145 E−9
40	0.4967294	5.214782 E−10	2.019321 E−10
45	0.4683210	3.458109 E−10	8.729451 E−10
50	0.4442883	1.248273 E−11	6.402321 E−11

Table 2. Numerical results of Example 2 in part I

N	h	$\left\|E^S_{\phi_1}(h)\right\|_\infty$	$\left\|E^S_{\phi_2}(h)\right\|_\infty$
5	1.404963	5.940656 E−3	1.474380 E−3
10	0.9934589	2.918275 E−4	2.361536 E−4
15	0.8111557	4.464388 E−5	1.257658 E−5
20	0.7024815	8.225441 E−6	2.920628 E−6
25	0.6283185	8.509960 E−7	7.152557 E−7
30	0.5735737	4.807629 E−8	5.960464 E−8
35	0.5310261	1.197832 E−8	3.135621 E−8
40	0.4967294	6.601731 E−9	4.047211 E−9
45	0.4683210	7.706242 E−10	2.942132 E−10
50	0.4442883	3.416235 E−11	7.066213 E−11

Table 3. Numerical results of Example 1 in part II in case of shifted Chebyshev polynomial method for N = 5:

x	Error
0.1	3.576279 E−7
0.2	3.576279 E−7
0.3	3.576279 E−7
0.4	2.384186 E−7
0.5	3.576279 E−7
0.6	3.576279 E−7
0.7	4.768372 E−7
0.8	4.768372 E−7
0.9	7.152557 E−7
1	8.344650 E−7

Table 4. Numerical results of Example 1 in part II in case of Sinc- collocation method

N	h	$\left\|E^S_{\varphi_1}(h)\right\|_\infty$
10	0.9934589	1.914620 E−3
15	0.8111557	2.176762 E−4
20	0.7024815	3.68356 E−5
25	0.6283185	7.748604 E−6
30	0.5735737	4.649162 E−6
35	0.5310261	9.536743 E−7
40	0.4967294	5.960464 E−7
45	0.4683210	4.053116 E−7
50	0.4442883	9.536743 E−7

Table 5. Numerical results of Example 2 in part II in case of shifted Chebyshev polynomial method for N = 5:

x	Error
0.0	9.164214 E−7
0.1	7.525086 E−7
0.2	5.811453 E−7
0.3	3.874302 E−7
0.4	2.235174 E−7
0.5	8.940697 E−8
0.6	2.980232 E−8
0.7	1.192093 E−7
0.8	2.086163 E−7
0.9	3.278255 E−7
1	4.619360 E−7

Table 6. Numerical results of Example 2 in part II in case of Sinc- collocation method

N	h	$\left\|E^S_{\varphi_1}(h)\right\|_\infty$
5	1.404963	7.003546 E−7
10	0.9934589	4.059984 E−9
15	0.8111557	5.820766 E−10
20	0.7024815	2.103206 E−12
25	0.6283185	0.000000000000
30	0.5735737	0.000000000000

Table 7. Numerical results of Example 3 in part II in case of shifted Chebyshev polynomial method for N = 5:

x	Error
0.0	1.490116 E−6
0.1	3.099442 E−6
0.2	2.175570 E−5
0.3	1.013279 E−5
0.4	1.642108 E−5
0.5	2.914667 E−5
0.6	1.484156 E−5
0.7	1.299381 E−5
0.8	2.312660 E−5
0.9	7.152557 E−7
1	3.099442 E−6

Table 8. Numerical results of Example 3 in part II in case of Sinc- collocation method

N	h	$\left\Vert E_{\varphi_1}^{S}(h)\right\Vert_\infty$
5	1.404963	1.708865 E−4
10	0.9934589	3.409386 E−5
15	0.8111557	1.549721 E−6
20	0.7024815	7.152557 E−7
25	0.6283185	5.960464 E−8
30	0.5735737	5.960464 E−8
35	0.5310261	0.000000000000

Conclusion

In part I of this paper we study the numerical solution of example 1 and example 2 by Sinc-collocation method. But example 1 has been studied by Taylor-series expansion method in [6] and example 2 has been studied by using Block-Pulse functions in [8] by comparing the results we find that our method is better than the results of Maleknejad et al.,[6] and Maleknejad et al.[8]. In part II we study the numerical solution of three examples of linear Fredholm integral equations by using shifted Chebyshev polynomial method and Sinc- collocation method which derive a good approximation.

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