Some approximation theorems via σ-convergence

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Abstract: The concept of σ-convergence was introduced in [P. Schaefer, Proc. Amer. Math. Soc. 36(1972)104-110] by using invariant mean.In this paper we apply this method to prove some Korovkin type approximation theorems.
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1. Introduction and preliminaries

Let c and ℓ_∞ denote the spaces of all convergent and bounded sequences, respectively, and note that $c \subset \ell_\infty$. In the theory of sequence spaces, a beautiful application of the well known Hahn-Banach Extension Theorem gave rise to the concept of the Banach limit. That is, the lim functional defined on c can be extended to the whole of ℓ_∞ and this extended functional is known as the Banach limit [2]. In 1948, Lorentz [8] used this notion of a weak limit to define a new type of convergence, known as the almost convergence. Later on, Raimi [17] gave a slight generalization of almost convergence and named it the σ-convergence. Before proceeding further, we should recall some notations and basic definitions used in this paper.

Let σ be a mapping of the set of positive integers $\mathbb{N}$ into itself. A continuous linear functional φ defined on the space ℓ_∞ of all bounded sequences is called an invariant mean (or a σ-mean; cf. [17]) if it is non-negative, normal and $\varphi(x) = \varphi\left(\left(x_{\sigma(n)}\right)\right)$.

A sequence $x = x_k$ is said to be *σ-convergent* to the number L if and only if all of its σ-means coincide with L, i.e. $\varphi(x) = L$ for all φ. A bounded sequence $x = x_k$ is σ-convergent (cf. [18]) to the number L if and only if $\lim_{p\to\infty} t_{pm} = L$ uniformly in m, where

$$t_{pm} = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p+1}$$

We denote the set of all σ-convergent sequences by V_σ and in this case we write $x_k \to L(V_\sigma)$ and L is called the σ-limit of x. Note that a σ-mean extends the limit functional on c in the sense that $\varphi = \lim x$ for all $x \in c$ if and only if σ has no finite orbits (cf. [11, 12]) and $c \subset V_\sigma \subset \ell_\infty$.

If σ is a translation then the σ-mean is called a Banach limit and σ-convergence is reduced to the concept of almost convergence introduced by Lorentz [8].

For σ-convergence of double sequences, we refer the reader to [3, 12, 13, 14].

If $m = 1$ then we get $(C,1)$; convergence, and in this case we write $x_k \to \ell(C,1)$; where $\ell = (C,1)$-lim x.

Remark 1.1. Note that:
(a) a convergent sequence is also σ-convergent;
(b) a σ-convergent sequence implies $(C,1)$ convergent.

Example 1.2. The sequence $z = (z_n)$ defined as

$$z_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

is σ-convergent to 1/2(for $\sigma(n) = n+1$) but not convergent.

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm $||f||_\infty := \sup_{a\le x\le b} |f(x)|, f \epsilon C[a,b]$. Suppose that $T_n : C[a,b] \to C[a,b]$. We write $T_n f(x)$ for $T_n(f(t),x)$ and we say that T is a positive operator if $T(f,x) \ge 0$ for all $f(x) \ge 0$.

The classical Korovkin approximation theorem states as follows [6, 7]:

Let T_n be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ and $\lim_n \|T_n(f_i, x - f_i(x)\|_\infty$=0, for $i = 0,1,2,$ where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$. Then $\lim_n \|T_n f(x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$.

Quite recently, such type of approximation theorems for functions of single variables were proved in [5, 9, 10, 15, 16] and for functions of two variables in [1, 4] by using statistical convergence and almost convergence. In this paper, we use the notion of σ-convergence to prove Korovkin type approximation theorems.

2. Korovkin type approximation theorem

The following is the V_σ-version of the classical Korovkin approximation theorem followed by an example to show its importance.\newline

Theorem 2.1. Let $(T)_k$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ and $D_{n,p}(f,x) = \frac{1}{p}\sum_{k=}^{p=1} T_{\sigma^k(n)} f(x)$ satisfying the following conditions

$$\lim_{p\to\infty}\left\|D_{n,p}(1,x)-1\right\|_\infty = 0 \quad \text{uniformly in } n, \qquad (2.1.1)$$

$$\lim_{p\to\infty}\left\|D_{n,p}(t,x)-x\right\|_\infty = 0 \quad \text{uniformly in } n, \qquad (2.1.2)$$

$$\lim_{p\to\infty}\left\|D_{n,p}(t^2,x)-x^2\right\|_\infty = 0 \quad \text{uniformly in } n, \qquad (2.1.3)$$

Then for any function $f \epsilon\, C[a,b]$ bounded on the whole real line, we have

$$\sigma\text{-}\lim_{k\to\infty}\|T_k(f,x)-f(x)\|_\infty = 0 \quad \text{i.e.,}$$

$$\lim_{p\to\infty}\left\|D_{n,p}(f,x)-f(x)\right\|_\infty = 0 \quad \text{uniformly in } n,$$

Proof. Since $f \epsilon\, C[a,b]$ and f is bounded on the real line, we have

$$|f(x)| \le M, \qquad -\infty < x < \infty.$$

Therefore,

$$|f(t)-f(x)| \le 2M, -\infty < t, x < \infty \qquad (2.1.4)$$

Also we have that f is continuous on $[a,b]$, i.e.,

$$|f(t)-f(x)| < c, \qquad \forall\, |t-x| < \delta \qquad (2.1.5)$$

Using (2.1.4), (2.1.5) and putting $\psi\,(t) = (t-x)^2$, we get

$$|f(t)-f(x)| < \epsilon + \frac{2M}{\delta^2}\psi, \forall |t-x| < \delta,$$

This means

$$-\epsilon - \frac{2M}{\delta^2}\psi < f(t)-f(x) < \epsilon + \frac{2M}{\delta^2}\psi.$$

Now, we operating $T_{\sigma^k(n)}(1,x)$ for all n to this inequality since $T_{\sigma^k(n)}(f,x)$ is monotone and linear. Hence

$$T_{\sigma^k(n)}(1,x)\left(-\epsilon - \frac{2M}{\delta^2}\psi\right) < T_{\sigma^k(n)}(1,x)\big(f(t)-f(x)\big) < T_{\sigma^k(n)}(1,x)\left(\epsilon + \frac{2M}{\delta^2}\psi.\right)$$

Note that x is fixed and so $f(x)$ is constant number. Therefore

$$-\epsilon T_{\sigma^k(n)}(1,x) - \frac{2M}{\delta^2}T_{\sigma^k(n)}(\psi,x) < T_{\sigma^k(n)}(f,x) - f(x)T_{\sigma^k(n)}(1,x) < \epsilon T_{\sigma^k(n)}(1,x) + \frac{2M}{\delta^2}T_{\sigma^k(n)}(\psi,x) \quad (2.1.6)$$

But

$$T_{\sigma^k(n)}(f,x) - f(x) = T_{\sigma^k(n)}(f,x) - f(x)T_{\sigma^k(n)}(1,x) + f(x)T_{\sigma^k(n)}(1,x) - f(x) = \left[T_{\sigma^k(n)}(f,x) - f(x)T_{\sigma^k(n)}(1,x)\right] + f(x)\left[T_{\sigma^k(n)}(1,x) - 1\right] \qquad (2.1.7)$$

Using (2.1.6) and (2.1.7), we have

$$T_{\sigma^k(n)}(f,x) - f(x) < \epsilon T_{\sigma^k(n)}(1,x) + \tfrac{2M}{\delta^2}T_{\sigma^k(n)}(\psi,x) + f(x)(T_{\sigma^k(n)}(1,x)-1)(2.1.8)$$

Let us estimate $T_{\sigma^k(n)}(\psi,x)$

$$\begin{aligned} T_{\sigma^k(n)}(\psi,x) &= T_{\sigma^k(n)}((t-x)^2,x) \\ &= T_{\sigma^k(n)}(t^2-2tx+x^2,x) \\ &= T_{\sigma^k(n)}(t^2,x) + 2xT_{\sigma^k(n)}(t,x) + x^2T_{\sigma^k(n)}(1,x) \\ &= [T_{\sigma^k(n)}(t^2,x) - x] - 2x[T_{\sigma^k(n)}(t,x) - x] \\ &\quad + x^2\left[T_{\sigma^k(n)}(1,x) - 1\right]. \end{aligned}$$

Using (2.1.8), we obtain

$$\begin{aligned} T_{\sigma^k(n)}(f,x) - f(x) &< \epsilon T_{\sigma^k(n)}(1,x) \\ &+ \frac{2M}{\delta^2}\Big\{\left[T_{\sigma^k(n)}(t^2,x) - x^2\right] + 2x\left[T_{\sigma^k(n)}(t,x) - x\right] \\ &\quad + x^2\left[T_{\sigma^k(n)}(1,x) - 1\right]\Big\} \\ &\quad + f(x)\big(T_{\sigma^k(n)}(1,x) - 1\big) \\ &= \epsilon\left[T_{\sigma^k(n)}(1,x) - 1\right] + \epsilon \\ &+ \frac{2M}{\delta^2}\Big\{\left[T_{\sigma^k(n)}(t^2,x) - x^2\right] + 2x\left[T_{\sigma^k(n)}(t,x) - x\right] \\ &\quad + x^2\left[T_{\sigma^k(n)}(1,x) - 1\right]\Big\} \\ &\quad + f(x)\big(T_{\sigma^k(n)}(1,x) - 1\big). \end{aligned}$$

Since ϵ is arbitrary, we can write

$$\begin{aligned} T_{\sigma^k(n)}(f,x) - f(x) &\le \epsilon\left[T_{\sigma^k(n)}(1,x) - 1\right] \\ &+ \frac{2M}{\delta^2}\Big\{\left[T_{\sigma^k(n)}(t^2,x)x^2\right] \\ &+ 2x\left[T_{\sigma^k(n)}(t,x) - x\right] \\ &+ x^2\left[T_{\sigma^k(n)}(1,x) - 1\right]\Big\} \\ &+ f(x)\big(T_{\sigma^k(n)}(1,x) - 1\big). \end{aligned}$$

Similarly

$$\begin{aligned} D_{n,p}(f,x) - f(x) &\le \epsilon\left[D_{n,p}(1,x) - 1\right] \\ &+ \frac{2M}{\delta^2}\Big\{\left[D_{n,p}(t^2,x) - x^2\right] \\ &+ 2x\left[T_{\sigma^k(n)}(t,x) - x\right] \\ &+ x^2\left[D_{n,p}(1,x) - 1\right]\Big\} \\ &+ f(x)\big(D_{n,p}(1,x) - x\big), \end{aligned}$$

and therefore

$$\left\|D_{n,p}(f,x) - f(x)\right\|_\infty \le \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right) \left\|D_{n,p}(1,x) - 1\right\|_\infty + \frac{4Mb}{\delta^2}\left\|D_{n,p}(t,x) - x\right\|_\infty + \frac{2M}{\delta^2}\left\|D_{n,p}(t^2,x) - x^2\right\|_\infty.$$

Letting $p \to \infty$ and using (2.1.1), (2.1.2), (2.1.3), we get

$$\lim_{p\to\infty}\left\|D_{n,p}(f,x)-f(x)\right\|_{\infty}=0 \text{ uniformly in } n$$

This completes the proof of the theorem.

In the following we give an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but does not satisfy the conditions of the Korovkin theorem.

Example 2.2.. Consider the sequence of classical Bernstein polynomials

$$b_n(f,x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \quad 0\le x\le 1.$$

Let the sequence (P_n) be defined by $P_n: C[0,1] \to C[0,1]$ with $P_n(f(,x) = (1+z_nB_n(f,x)$, where z_n is defined as in Example 1.2. Then

$$B_n(1,x)=1, B_n(t,x)=x, B_n(t^2,x)=x^2+\frac{x-x^2}{n},$$

and the sequence (P_n) satisfies the conditions (2.1.1)-(2.1.3). Hence we have

$$\sigma\text{-lim } \|P_n(f,x)-f(x)-f(x)\|_{\infty}=0.$$

On the other hand, we get $\$P_n(f,0)=(1+z_n)f(0)$, since $B_n(f,0)=f(0)$, and hence

$$\|P_n(f,x)-f(x)\|_{\infty} \ge |P_n(f,0)| = z_n|f(0)|$$

We see that (P_n) does not satisfy the classical Korovkin theorem, since $\lim\ \sup_{n\to\infty} z_n$ does not exists.

Now we present a slight general results.

Theorem 2.3. Let T_n be a sequence of positive linear operators on $C[a,b]$ such that

$$\lim_n\|T_{n+1}-T_n\|_{\infty}=0 \qquad (2.3.1)$$

If

$$\sigma\text{-}\lim_n\|T_n(t^{\nu}-x)-x^{\nu}\|_{\infty} = 0 \quad (\nu=0,1,2). \qquad (2.3.2)$$

Then for any function $f \in C[a,b]$ bounded on the real line, we have

$$\lim_n\|T_n(f,x)-f(x)\|_{\infty}=0 \qquad (2.3.3)$$

Proof. From Theorem 2.1, we have that if (2.3.2) holds then

$$\lim_p\left\|D_{n,p}(f,x)-f(x)\right\|_{\infty} = 0, \text{uniformly in } n \qquad (2.3.4)$$

We have the following inequality

$$\|T_n(f,x)-f(x)\|_{\infty} \le \left\|D_{n,p}(f,x)-f(x)\right\|_{\infty} + \frac{1}{p}\sum_{k=n+1}^{n+p-1}\left(\sum_{l=n+1}^{k}\|T_l-T_{l-1}\|_{\infty}\right)$$

$$\le \left\|D_{n,p}(f,x)-f(x)\right\|_{\infty} + \frac{p-1}{2}\left\{\sup_{k\ge n}\|T_k - T_{k-1}\|_{\infty}\right\} \qquad (2.3.5)$$

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Hence using (2.3.1) and (2.3.4), we get (2.3.3).

This completes the proof of the theorem.

Remark 2.4. We know that σ-convergence implies $(C,1)$ convergence. This motivates us to further generalize our main result by weakening the hypothesis or to add some condition to get more general result.

Theorem 2.5. Let $(T_n$ be a sequence of positive linear operators on $C[a,b]$ such that

$$(C,1)-\lim_{n\ge p}\|T_n(t^{\nu},x)-x^{\nu}\|_{\infty} = 0 \quad (\nu=0,1,2) \qquad (2.5.1)$$

and

$$\lim_p\left\{\sup_{n\ge p}\frac{n}{p}\left\|\xi_{n+p-1}(f,x)-\xi_{n-1}(f,x)\right\|_{\infty}\right\} = 0 \quad (2.5.2)$$

where

$$\xi_n(f,x)=\frac{1}{n+1}\sum_{k=0}^{n}T_k(f,x).$$

Then for any function $f \in C[a,b]$ bounded on the real line, we have

$$\sigma \lim_{n\to\infty}\|T_n(f,x)-f(x)\|_{\infty}=0,$$

Proof. For $n \ge p \ge 1$, it is easy to show that

$$D_{n,p}(f,x)=\xi_{n+p-1}(f,x) + \frac{n}{p}\left(\xi_{n+p-1}(f,x)-\xi_{n-1}(f,x)\right),$$

which implies

$$\sup_{n\ge p}\left\|D_{n,p}(f,x)-\xi_{n+p-1}(f,x)\right\|_{\infty} = \sup_{n\ge p}\frac{n}{p}\left\|\xi_{n+p-1}(f,x) - \xi_{n-1}(f,x))\right\|_{\infty} \quad (2.5.3)$$

Also by Theorem 2.1, Condition (2.5.1) implies that

$(C,1)\text{-}\lim_{n\to\infty}\|T_n(f,x)-f(x)\|_{\infty}=0$ (2.5.4)

Using (2.5.1)-(2.5.4) and the fact that σ-convergence implies $(C,1)$ convergence, we get the desired result.

This completes the proof of the theorem.

Theorem 2.6. Let $(T_n$ be a sequence of positive linear operators on $C[a,b]$ such that

$$\lim_n\sup_m\frac{1}{n}\sum_{k=0}^{n-1}\left\|T_n-T_{\sigma^k(m)}\right\|=0$$

If

$$\sigma\text{-}\lim_n\|T_n(t^{\nu},x-x^{\nu}\|_{\infty}=0 \quad (\nu=0,1,2) \quad (2.6.1)$$

Then for any function $f \in C[a,b]$ bounded on the real line, we have

$$\lim_n \|T_n(f,x) - f(x)\|_\infty = 0. \qquad (2.6.2)$$

Proof. From Theorem 2.1, we have that if (2.6.1) holds then

$$\sigma\text{-}\lim_n \|T_n(f,x) - f(x)\|_\infty = 0,$$

which is equivalent to

$$\lim_n \left\| \sup_m D_{m,n} = (f,x) - f(x) \right\|_\infty = 0$$

Now

$$T_n - D_{m,n} = T_n - \frac{1}{n}\sum_{k=0}^{n-1} T_{\sigma^k(m)}$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} (T_n - T_{\sigma^k(m)}).$$

Therefore

$$T_n - \sup_m D_{m,n} = \sup_m \frac{1}{n}\sum_{k=0}^{n-1} (T_n - T_{\sigma^k(m)}).$$

Hence using the hypothesis we get

$$\lim_n \|T_n(f,x) - f(x)\|_\infty$$
$$= \lim_n \left\| \sup_m D_{m,n}(f,x) - f(x) \right\|_\infty$$
$$= 0,$$

that is (2.6.2) holds.

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