Direct method for solving $1D$ convection-diffusion by block pulse functions with error analysis

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Abstract: In this paper, we introduce new numerical method to deal with convection-diffusion problem.The proposed method is based on two dimensional block pulse functions under the framework of projection method.In this approach, we use operational matrices instead of partial derivatives, thus any PDEs problem is converted to linear or nonlinear system of Algebra. Error analysis for this method are given. Numerical examples demonstrate the efficiency and accuracy of this method.

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1. Introduction

A stochastic process $X(t)$, associated with the convection-diffusion equation obey the stochastic differential equation

$$dX + V(X)dt = \sqrt{2D}dW, \quad (1)$$

where dW is the differential of a wiener process with unit variance. The above stochastic equation can be solved by finite difference method [7]. Although stochastic method do not suffer from the numerical diffusion on grid-based methods, they typically lose accuracy in the vicinity of interfacial boundaries. By Feymann-Kac theorem, equation (1) convert to $1D$-convection-diffusion equation [7]

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\varepsilon\frac{\partial^2 u}{\partial x^2}, \quad 0\le x\le 1, t\ge 0 \qquad (2)$$

subject to the initial and boundary conditions:

$$u(x,0) = f(x), \quad 0 \le x \le 1,$$
$$u(0,t) = g_1(t), \qquad t \ge 0, \qquad (3)$$
$$u(1,t) = g_2(t). \qquad t \ge 0.$$

The convection-diffusion equation is a parabolic partial differential equation combining the diffusion equation and the advection equation, which describes physical phenomena where particles system due to two processes: diffusion and convection.Convection refers to the movement of a substance within a medium (e.g., water or air). Diffusion is the movement of the substance from an area of high concentration to an area of low concentration, resulting in the uniform distributed of the substance[2].

The numerical methods for solving convection-diffusion model have been an active subject of research during the last thirty years [1, 4, 15]. The development of the new techniques which can solve the model still attract substantial attention. Numerical grid-based methods such as the finite element method (FEM) [6, 16], the finite difference method (FDM) [9, 10, 11], the finite element method (FVM) [14] and spectral method [18, 19], were widely applied these last decades and remain most popular. However, the methods suffer from some limitations (difficulties on irregular or complex geometry and on mesh distortion or large deformation problems).This paper led to the development of a methods based on series expansion and piecewise constant functions.

In this paper, we use two dimensional block pulse functions $(2DBPFs)$ to solve convection-diffusion model.By applying $(2DBPFs)$ based on direct method, any PDEs convert to linear or nonlinear system of Algebra. We use operational matrices for partial derivatives. This method is simple and it's applicable for any PDEs.

An outline of the paper is as follows:In section 2, we introduce $2DBPFs$ and their properties.In section 3 we present operational matrices for partial derivatives. Direct method for solving convection-diffusion equation are given in section 4. Error analysis for proposed method are investigated in section 5. Finally, in section 6,we apply the proposed method on some examples showing the accuracy and efficiency of the method.

2. Two dimensional Block-Pulse functions

A set of two dimensional Block-Pulse functions $\Phi_{i_1,i_2}(x,t)(i_1=0,1,2,\ldots,m_1-1,\, i_2=0,1,2,\ldots,m_2)$ is defined in the region $x\in[a,b)$ and $t\in[0,T)$ as:

$$\phi_{i_1,i_2}(x,t)=\begin{cases}1 & (i_1)h_1\le x<(i_1+1)h_1,(i_2)h_2\le t<(i_2+1)h_2\\ 0 & otherwise.\end{cases} \quad (4)$$

where m_1, m_2 are arbitrary positive integers, and $h_1=\frac{b-a}{m_1}, h_2=\frac{T}{m_2}$. There are some properties for $2DBPFs$ as following:

The $2DBPFs$ are disjoint, orthogonal and complete set $[3,5,12,13]$.

We can also expand a two variable function $u(x,t)$ into $BPFs$ series:

$$u(x,t)\cong\sum_{i_1=0}^{m_1-1}\sum_{i_2=0}^{m_2-1}u_{i_1,i_2}\phi_{i_1,i_2}(x,t), \quad (5)$$

through determining the block pulse coefficients:

$$u_{i_1,i_2}=\frac{1}{h_1h_2}\int_{(i_1)h_1}^{(i_1+1)h_1}\int_{(i_2)h_2}^{(i_2+1)h_2}u(x,t)dxdt, \quad (6)$$

Also, for vector forms, consider the m^2 elements of $2DBPFs$

$$\Phi(x,t)=[\phi_{0,0},\phi_{0,1},\ldots,\phi_{0,m-1},\ldots,\phi_{m-1,0},\ldots,\phi_{m-1,m-1}]^T(x,t). \quad (7)$$

The two important properties of 2DBPF's are given as

(i):

$$\Phi(x,t)\Phi^T(x,t)V=\tilde{V}\Phi(x,t), \quad (8)$$

where V is an m^2 vector and $\tilde{V}=diag(V)$. Moreover, it can be clearly concluded that for every $m^2\times m^2$ matrix B:

(ii):

$$\Phi^T(x,t)B\Phi(x,t)=\hat{B}^T\Phi(x,t), \quad (9)$$

where $\hat{B}$ is an m^2 column vector with elements equal to the diagonal entries of matrix B. For simplicity, we use $m_1=m_2=m$.

Let $D_T=\{(x,t): a<x<b, 0<t<T\}$, where $-\infty\le a<b\le\infty$, and $\partial_p D_T$ be the parabolic boundary of D_T. If a,b are finite,

$$\partial_p D=\{x=a, x=b, 0\le t\le T\}\cup\{a\le x\le b, t=0\},$$

If a,b are infinite,

$$\partial_p D=\{x\in\Re, t=0\}$$

and

$$L^{2,1}(D_T)=\{u(x,t): u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2}\in L^2(D_T)\}. \quad (10)$$

without loss of generality, set $a=0, b=1$ and $T=1$. The inner product $<.,.>$ and norm $\|.\|$ in $L^{2,1}(D_T)$ are defined as follows:

$$<u(x,t),v(x,t)>=\int_0^1\int_0^1 u(x,t)v(x,t)dxdt, \quad (11)$$

$$\|u(x,t)\|=\left(\int_0^1\int_0^1 u^2(x,t)dxdt\right)^{\frac{1}{2}}. \quad (12)$$

Let P_m be the projection operator defined on $L^{2,1}(D_T)\to\mathbb{B}$, where $\mathbb{B}$ is finite m^2 −dimensional, as:

$$u_m(x,t)=P_m u(x,t)=\sum_{j=0}^{m-1}\sum_{i=0}^{m-1}u_{i,j}\phi_{i,j}(x,t). \quad (13)$$

First, we find an estimation of $\|u-P_m u\|$ for arbitrary $u\in L^{2,1}(D_T)$.

Lemma 1 Let $u(x,t)$ be defined on $L^{2,1}(D_T)$ and P_m be projection operator defined by (13) then

$$\|u-P_m u\|\le\frac{max|u|}{2\sqrt{3}m}, \quad (14)$$

where $\max|u|=\max_{0\le i,j\le m-1}|u_{ij}|$ for $0\le i,j\le m-1$.

Proof: The integral $\int_0^t\int_0^1 u_{i,j}\Phi(x,y)dxdy$ is a ramp $\frac{u_{i,j}}{m}(t-\frac{i}{m})$ on the subinterval $[\frac{i}{m},\frac{i+1}{m})\times[\frac{j}{m},\frac{j+1}{m})$ with average value $\frac{u_{i,j}}{2m^2}$.

The error in approximating the ramp by this constant value over the subinterval $[\frac{i}{m},\frac{i+1}{m})\times[\frac{j}{m},\frac{j+1}{m})=I_{ij}$ is

$$r_{i,j}(s,t)=\frac{u_{i,j}}{2m^2}-\frac{u_{i,j}}{m}\left(t-\frac{i-1}{m}\right), \quad (15)$$

hence, using E_{ij} as least square of the error on I_{ij}, we have

$$E_{i,j}^2=\int_{\frac{j-1}{m}}^{\frac{j}{m}}\int_{\frac{i-1}{m}}^{\frac{i}{m}}(r_{i,j}(s,t))^2dsdt\le\frac{|u_{i,j}|^2}{12m^6}, \quad (16)$$

$$E_{i,j}\le\frac{|u_{i,j}|}{2\sqrt{3}m^3}, \quad (17)$$

and on the interval D_T we have

$$\|u-P_m u\|=maxE_{ij}\le\frac{max|u|}{2\sqrt{3}m}. \quad (18)$$

Operational matrix for partial derivatives

The expansion of function $u(x,t)$ over D_T with respect to $\phi_{i,j}(x,t), i,j = 0,1,\dots m-1$, can be written as

$$u(x,t) \cong \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} u_{i,j}\phi_{i,j}(x,t) = U^T\Phi = \Phi^T U, \tag{19}$$

Where

$$U = [u_{0,0}, u_{0,1}, \dots, u_{0,m-1}, u_{1,0}, \dots, u_{1,m-1}, \dots, u_{m-1,m-1}]^T,$$

$$\Phi = [\phi_{0,0}, \phi_{0,1}, \dots, \phi_{0,m-1}, \phi_{1,0}, \dots, \phi_{1,m-1}, \dots, \phi_{m-1,m-1}]^T$$

, and

$$\phi_{i,j}(x,t) = \begin{cases} 1 & \frac{i}{m} \le x < \frac{i+1}{m}, \frac{j}{m} \le t < \frac{j+1}{m} \\ 0 & otherwise, \end{cases} \tag{20}$$

$$u_{ij} = \frac{1}{h^2}\int_{\frac{j}{m}}^{\frac{j+1}{m}}\int_{\frac{i}{m}}^{\frac{i+1}{m}} u(x,t)dxdt. \tag{21}$$

Now, expressing $\int_0^1\int_0^t \phi_{i,j}(s,y)dsdy$, in terms of the $2DBPFs$ as :

$$\int_0^1\int_0^t \phi_{i,j}(s,y)dsdy \cong [0,0,\dots,0,\frac{h^2}{2},h^2,\dots,h^2], \tag{22}$$

in which $\frac{h^2}{2}$, is ith component. Thus

$$\int_0^1\int_0^t \Phi(s,y)dsdy \cong P\Phi(x,t), \tag{23}$$

where P is $m^2 \times m^2$ matrix and is called operational matrix of double integration and can be denoted by $P = \frac{h^2}{2}P_2$, where

$$\mathbf{P}_2 = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{24}$$

so, the double integral of every function $u(x,t)$ can be approximated by:

$$\int_0^1\int_0^t u(s,y)dsdy \cong \frac{h^2}{2}U^T P_2\Phi(x,t), \tag{25}$$

by similar method $\int_0^1 \Phi_{ij}(s,t)ds$, in terms of $2DBPFs$ as:

$$\int_0^1 \phi_{ij}(s,t)ds \cong [0,0,\dots,h,0,0,\dots,0]^T\Phi(.,.), \tag{26}$$

and

$$\int_0^1 \Phi(s,t)ds \cong hI\Phi(.,.). \tag{27}$$

Now, we compute operational matrix for $\frac{\partial u}{\partial t}$

Lemma 2 Suppose $u \in L^{2,1}(D_T)$ and u is defined on parabolic boundary $\partial_p D_T$ then operational matrix for $\frac{\partial u}{\partial t}$ by 2DBPFs is approximated as:

$$\frac{\partial u(x,t)}{\partial t} \cong (U_t^d)^T\Phi(x,t) \tag{28}$$

that:

$$U_t^d = \frac{2}{h}(U^T - U_f^T\Delta_1)P_2^{-1}, \tag{29}$$

where Δ_1 is the following $m^2 \times m^2$ matrix as:

$$\Delta_1 = \begin{pmatrix} H_{m\times m} & & 0 \\ & H_{m\times m} & \\ 0 & & \ddots \\ & & & H_{m\times m} \end{pmatrix}, \tag{30}$$

with

$$\mathbf{H}_{m\times m} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{31}$$

that U_f is initial boundary vector of $\partial_p D_T$.

Proof: By applying approximation $\frac{\partial u}{\partial t} \cong (U_t^d)^T\Phi$ in (23) instead of u we have:

$$\int_0^1\int_0^t \frac{\partial u(s,y)}{\partial y}dsdy \cong \frac{h^2}{2}(U_t^d)^T P_2\Phi(x,t), \tag{32}$$

and

$$\begin{aligned}\int_0^1\int_0^t \frac{\partial u(s,y)}{\partial y}dsdy &= \int_0^1 (u(s,t) - u(s,0))ds \\ &= \int_0^1 (U^T\Phi(s,t) - U_f^T\Phi(s,0))ds \\ &= hU^T I\Phi(x,t) - hU_f^T\Delta_1\Phi(x,t),\end{aligned} \tag{33}$$

from (32) and (33) we can conclude:

$$U_t^d = \frac{2}{h}(U^T - U_f^T\Delta_1)P_2^{-1}. \tag{34}$$

by the same method, operational matrix for $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ are given as follows.

Lemma 3 If $u \in L^{2,1}(D_T)$ and defined in parabolic boundary $\partial_p D_T$ then operational matrix for $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ by 2DBPFs are approximated as:

$$\frac{\partial u}{\partial x} \cong (U_x^d)^T \Phi(x,t), \quad (35)$$

$$\frac{\partial^2 u}{\partial x^2} \cong (U_{xx}^d)^T \Phi(x,t), \quad (36)$$

where

$$U_x^d = \frac{1}{h}(U_{g_2}^T \Delta_3 - U_{g_1}^T \Delta_2) P_2^{-1}, \quad (37)$$

$$U_{xx}^d = \frac{1}{h^2}(U_{g_2}^T \Delta_3 - U_{g_1}^T \Delta_2) P_2^{-1}(\Delta_3 - \Delta_2) P_2^{-1}, \quad (38)$$

and Δ_2, Δ_3 are the following $m^2 \times m^2$ matrices:

$$\Delta_2 = \begin{pmatrix} I_{m\times m} & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}, \quad (39)$$

$$\Delta_3 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & I_{m\times m} \end{pmatrix}, \quad (40)$$

and U_{g_1}, U_{g_2} are boundary vectors of $\partial_p D_T$.

Direct method for solving nonlinear *PDEs*

The results obtained in previous section are used to introduce a direct efficient and simple method to solve equations $(2)-(3)$. We consider equations $(2)-(3)$ of the form:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, (x,t) \in D_T \quad (41)$$

$$u(x,0) = f(x), \quad u(0,t) = g_1(x), \quad u(1,t) = g_2(x). \quad (42)$$

By substituting the equations (29), (37) and (38) into (41) and using boundary and initial conditions, we obtain a linear system with $u_{i,j} (i,j = 0,1,\dots,m-1)$ as unknowns:

$$(U_t^d)^T - \varepsilon((U_{xx}^d)^T) + c(U_x^d)^T = 0. \quad (43)$$

Error analysis

Let the problem be of the form

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, (x,t) \in D_T, \quad u(x,0) = f(x), \quad u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \quad (44)$$

where $f(x)$, $g_1(t)$, $g_2(t)$ belong to $L^2[0,1)$.

By using (13), the discrete approximation of (41) is :

$$\frac{\partial u_m}{\partial t} + c\frac{\partial u_m}{\partial x} = \varepsilon \frac{\partial^2 u_m}{\partial x^2} + e, \quad (45)$$

where, for each (x,t), $P_m u(x,t)$ belongs to an m^2 —dimensional subspace $\mathbb{B}$.

Theorem 1 Let $u(x,t)$ and $f(x,t)$ be in $L^{2,1}(D_T)$ and $u_m(x,t)$ be approximate solution by 2DBPFs of (13)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad P_m \frac{\partial u}{\partial t} + cP_m \frac{\partial u}{\partial x} = \varepsilon P_m \frac{\partial^2 u}{\partial x^2} + e, \quad (46)$$

then

$$\|e\| \le \frac{1}{2\sqrt{3}m}\left(\max\left|\frac{\partial u}{\partial t}\right| + \varepsilon\max\left|\frac{\partial^2 u}{\partial x^2}\right| + c\max\left|\frac{\partial u}{\partial x}\right|\right) \quad (47)$$

Proof: By using properties of projection operators,

$$e = \frac{\partial u(x,t)}{\partial t} - P_m\frac{\partial u(x,t)}{\partial t} - \varepsilon\left(\frac{\partial^2 u(x,t)}{\partial x^2} - P_m\frac{\partial^2 u(x,t)}{\partial x^2}\right) + c\left(\frac{\partial u(x,t)}{\partial x} - P_m\frac{\partial u(x,t)}{\partial x}\right) \quad (48)$$

$$\|e\| \le \left\|(I-P_m)\frac{\partial u(x,t)}{\partial t}\right\| + \varepsilon\left\|(I-P_m)\frac{\partial^2 u(x,t)}{\partial x^2}\right\| + c\left\|(I-P_m)\frac{\partial u(x,t)}{\partial x}\right\| \quad (49)$$

$$\|e\| \le \frac{1}{2\sqrt{3}m}\left(\max\left|\frac{\partial u}{\partial t}\right| + \varepsilon\max\left|\frac{\partial^2 u}{\partial x^2}\right| + c\max\left|\frac{\partial u}{\partial x}\right|\right) \quad (50)$$

$$\|e\| \le \frac{A}{2\sqrt{3}m}$$

where $A = \max\left|\frac{\partial u}{\partial t}\right| + \varepsilon\max\left|\frac{\partial^2 u}{\partial x^2}\right| + c\max\left|\frac{\partial u}{\partial x}\right|$ for $(x,t) \in D_T$, so by hypothesis of the theorem, A is a finite number and $\|e\| = O(\frac{1}{m})$. So, if $m \to \infty$ then $\|e\|$ tends to zero.

Numerical example

We present results of some numerical experiments to illustrate the effectiveness of the proposed method. To this end we choose convection-diffusion equations taken from (Khojasteh Salkuyeh

2006) which are characterized by the fact of having parameter dependent solutions of the form

$$u(x,t) = \exp(\alpha x + \beta t), 0 \le x \le 1, t \ge 0,$$

where α, β are adjusted such that the condition $\varepsilon^2 - c\alpha - \beta = 0$ is satisfied.Initial and boundary conditions are in the case

$$u(x,0) = f(x) = \exp(\alpha x), u(0,t) = \exp(\beta t) = g_1(t), u(1,t) = g_2(t) = \exp(\alpha + \beta t).$$

To show the efficiently of the present method we report absolute error which is defined by

$$e_{i,j} = \left\| u(x_i,t_j) - u_m\,(x_i,t_j) \right\|, \quad (51)$$

at the point (x_i,t_j) where $u(x_i,t_j)$ is exact solution and $u_{i,j}$ is numerical solution by 2DBPFs. Error surface are plotted for showing the accuracy for two examples.

Example 1 Parameters defining the problem (2)-(3) and the corresponding solution are

$$c = 3.5, \varepsilon = .022, \alpha = .02854797991928, \beta = -.0999.$$

Example 2 Parameters defining the problem (2)-(3) and the corresponding solution are

$$c = 1, \varepsilon = .09, \alpha = .001, \beta = -.00099.$$

Table 1: Error between exact and numerical solution for example 1 (m=10)

x_i	t_j	Exact solution	Numerical solution	Error
0.0	0.0	9.9279e-1	9.8332e-1	9.4636e-3
0.0	0.2	9.7372e-1	9.6366e-1	9.5508e-3
0.0	0.4	9.5392e-1	9.4455e-1	9.3617e-3
0.4	0.0	1.0072e-0	9.9972e-1	9.9715e-3
0.4	0.2	9.6771e-1	9.7736e-1	9.8724e-3
0.4	0.4	9.6771e-1	9.5803e-1	9.6771e-3
0.7	0.0	1.0158e-0	1.0057e-0	1.0058e-2
0.7	0.2	9.9573e-1	9.8576e-1	9.9573e-3
0.7	0.4	9.7604e-1	9.6627e-1	9.7603e-3

Table 2: Error between exact and numerical solution for example 2 (m=10)

x_i	t_j	Exact solution	Numerical solution	Error
0.0	0.0	1.000e-0	9.9015e-1	9.8050e-3
0.0	0.2	9.9980e-1	9.8989e-1	9.9030e-3
0.0	0.4	9.9660e-1	9.8969e-1	9.9010e-3
0.4	0.0	1.0005e-0	9.9059e-1	9.9050e-3
0.4	0.2	1.0003e-0	9.9030e-1	1.0003e-2
0.4	0.4	1.0001e-0	9.9010e-1	1.0001e-2
0.7	0.0	1.0003e-0	9.9039e-1	9.9080e-3
0.7	0.2	1.0001e-0	9.9010e-1	1.0006e-2
0.7	0.4	9.9991e-1	9.8990e-1	1.0004e-2

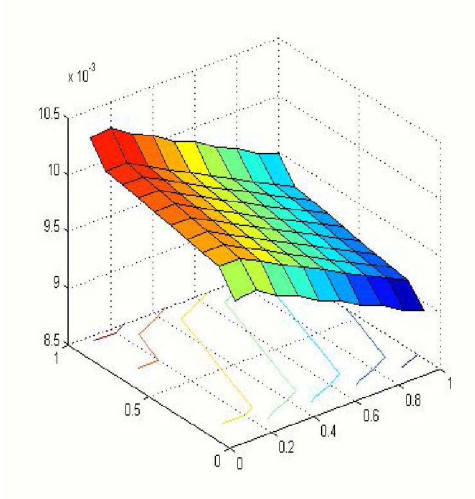

Figure 1: Error surface for example 1

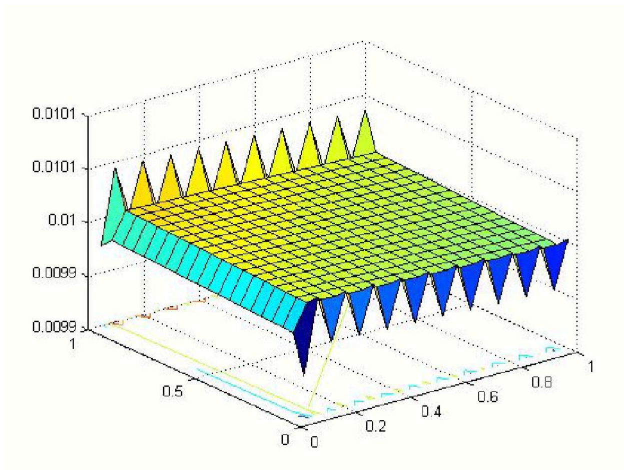

Figure 2 : Error surface for example 2

3. Discussion

In this paper, we introduced a new numerical scheme for convection diffusion equation by two dimension block pulse functions and their operational matrices for partial derivatives. This method can be used for any linear and nonlinear partial differential equations. We can say that this method is feasible and the error is acceptable. the implementation of the present method is a very easy, acceptable and valid. We can use other piecewise constant functions for example Haar, Walsh and wavelets.

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