Singularities of Gauss Map of Pedal Hypersurface in R^{n+1}

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Abstract: This paper mainly studies the singularities of Gauss Map of pedal hypersurface in R^{n+1}. It contains the geometry of pedal hypersurfaces in R^{n+1} and their Gauss maps. The singularity of Gauss map of the pedal hypersurface using the rank of jacobian matrix of Gauss map is given and classified. The sets of singularities and its graphs under the Gauss map are plotted.
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1. Geometry of Pedal Hypersurfaces in Euclidean Space

In this section we review the classical theory of differential geometry on hypersurfaces in Euclidean space R^{n+1} [1],[2],[3].

Let $X : U \to R^{n+1}$ be an embedding and U is an open subset of R^n, and identify M and U through the embedding X , i.e., M = X (U), in this case M is called hypersurface in R^{n+1} . The tangent space of M at p = X (u), u ∈ U is

$$T_pM = \langle X_1(u), X_2(u), ..., X_n(u)\rangle, \quad X_i = \frac{\partial X}{\partial u_i} \qquad (1)$$

and the unit normal vector field along $X : U \to R^{n+1}$ is given by:

$$N(u) = \frac{X_1(u) \times X_2(u) \times ... \times X_n(u)}{\|X_1(u) \times X_2(u) \times ... \times X_n(u)\|} \qquad (2)$$

Where

$$X_1 \times X_2 \times ... \times X_n = \begin{vmatrix} e_1 & e_2 & \cdots & e_{n+1} \\ X_1^1 & X_1^2 & \cdots & X_1^{n+1} \\ X_2^1 & X_2^2 & \cdots & X_2^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ X_n^1 & X_n^2 & \cdots & X_n^{n+1} \end{vmatrix}$$

where $\{e_1 , \ldots, e_{n+1}\}$ is the canonical basis of R^{n+1} and

$$X_i = X_i^j \in T_pM \subset R^{n+1}, \ X_i^j = \frac{\partial X_i}{\partial u_j}, \ X = (X_i)$$

A map $G : U \to S^n$ defined by G (u) = N (u) is called the Gauss map of M =X (U), the derivative of the Gauss map $dG (u) : T_p M \to T_p M$ can be interpreted as a liner transformation on the tangent space T p M .The linear transformation S p = -dG (u) is called the shape operator (or Weingarten map) of the hypersurface M = X (U) . The eigenvalues of S_p are called the principal curvatures. and the eigenvectors of S_p are called the principal directions on M . By definition, k_p is a principal curvature if and only if det (S_p - k_p I) = 0. The Gauss-Kronecker curvature of M = X (U) at p = X (u) is defined to be K (u) = $\det S_p$.

In the extrinsic differential geometry, totally umbilical hypersurfaces are considered to be the model hypersurfaces in Euclidean space. Since the set $\{ X_i|\ (i = 1, \ldots, n)\}$ is linearly independent, the Riemannian metric (first fundamental form) on M = X (U)is given by $ds^2 = \sum_{i=1}^{n} g_{ij}\, du_i\, du_j$, where $g_{ij} = \langle X_i(u), X_j(u)\rangle$ for any u ∈ U . The second fundamental coefficients l_{ij} are given by $l_{ij} = \langle -N_i (u), X_j (u)\rangle = \langle N (u), X_{ij} (u)\rangle$, for any u ∈ U. Recall the following Weingarten formula [4]:

$$N_i(u) = -l_i^j(u)X_j(u) \qquad (3)$$

Where $l_i^j(u) = l_{ik}(u)g^{kj}(u), g^{kj}(u) = (g_{kj}(u))^{-1}$ and $g_{ik}(u)g^{kj}(u) = \delta_i^j$.

By the Weingarten formula, the Gauss-Kronecker curvature is given by (4)

$$K(u) = \frac{\det((l_{ij})(u))}{\det((g_{\alpha\beta})(u))}$$

For a hypersurface $X: U \to R^{n+1}$, the pedal hypersurface of M = X (U) is $\hat{M}$ and defined by :

$$\hat{X} : U \to R^{n+1}; \hat{X}(u) = \langle X(u), N(u)\rangle N(u) = S(u)N(u)$$

where S (u) =< X (u) , N (u) > ≠ 0 is the support function on M and $\hat{M} = \hat{X}(U)$ is identified with U through the embedding $\hat{X}$ [3]. The tangent space of $\hat{M}$ at $\hat{p} = \hat{X}(u)$ is defined as:

$\hat{T}_{\hat{p}}(\hat{M})$

The unit normal vector field along $\hat{X}$: U → R^{n+1} can b e obtained and is given by:

$$\hat{N}(u) = (-1)^n \frac{2\hat{X}(u) - X(u)}{\|X(u)\|} \tag{6}$$

Since the set { $\hat{X}_i$ } is linearly independent, the Riemannian induce metric on $\hat{M} = \hat{X}(U)$ (first fundamental form) is given as:

$$ds^2 = \sum_{i=1}^n \hat{g}_{ij} du_i du_j = \sum_{i=1}^n (S_{u_i} S_{u_j} + S^2 \eta_{ij}) du_i du_j$$

where $\hat{g}_{ij} = \langle \hat{X}_i(u), \hat{X}_j(u) \rangle$ and $\eta_{ij} = \langle N_i(u), N_j(u) \rangle$ is the third fundamental metric for any u ∈ U .The discriminant $\hat{g}$ on pedal hypersurface $\hat{M}$ is given by the relation:

$$\hat{g} = det(\hat{g}_{ij}) = K^2(u) g(u) \|X(u)\|^2 S^{2n-2} = K(u) l(u) \|X(u)\|^2 S^{2n-2} \tag{7}$$

where g = det(g_{ij}), l = det(l_{ij}) and K (u) is the Gaussian curvature of the hypersurface M .

The second fundamental co efficients l_{ij} are given by:

$$\hat{l}_{ij} = \langle -\hat{N}_i(u), X_j(u) \rangle = \frac{(-1)^{n+1}}{\|X(u)\|}(2\hat{g}_{ij} + Sl_{ij}), \quad \hat{N}_i = \frac{\partial \hat{N}}{\partial u_i}$$

for any u ∈ U . Thus Weingarten formula on $\hat{M}$ is given as:

$$\hat{N}_i(u) = -\bar{l}_i^j(u) \hat{X}_{u_j}(u) = \frac{(-1)^{n+2}}{\|X(u)\|}\left(2\delta_{ij} + S(u) l_{ik}(u) \hat{g}^{kj}(u)\right) \tag{8}$$

Where

$$\ddot{l}_i^j(u) = \hat{l}_{ik}(u) \hat{g}^{kj}(u) \ \ and \ \ \hat{g}^{kj}(u) = (\hat{g}_{kj}(u))^{-1} \quad .$$

From (8) it is easy to see that, the Gauss-Kronecker curvature $\hat{K}$ is given by:

$$\hat{K}(u) = \frac{\det(\hat{l}_{ij})}{\det(\hat{g}_{\alpha\beta})} \tag{9}$$

Explicitly the Gauss-Kronecker $\hat{K}$ curvature can be written as:

$$\hat{K}(u) = \frac{1}{\|X(u)\|^n} \prod_{i=1}^n \left(2\delta_{ij} + S(u) l_{ik}(u) \hat{g}^{kj}(u)\right) \tag{10}$$

and the mean curvature is given by:

$$\hat{H}(u) = \frac{(-1)^{n+1}}{nK(u)\|X(u)\|^3 S(u)^{2n-3}}$$

$$\left(2nK(u)\|X(u)\|^2 S(u)^{2n-3} + S_i^2 l^{ii} + nS(u)^2 H(u)\right)$$

where H(u) is the mean curvature of the hypersurface M at u ∈ U. The third fundamental coefficients are given by:

$$\hat{\eta}_{ij} = \frac{(4\hat{g}_{ij}(u) + g_{ij}(u))}{\|X(u)\|^2} + \frac{(\langle X(u), X_i(u)\rangle\langle X(u), X_j(u)\rangle)}{\|X(u)\|^4} \tag{11}$$

By the definition the point q = $\hat{X}(u) \in \hat{M}$ is a parabolic point if $\hat{K}(u) = 0$. Thus the hyperbolic set on $\hat{M}$ is define by (using eq. (10)) :

$$\prod_{i=1}^n \left(2\delta_{ij} + S(u) l_{ik}(u) \hat{g}^{kj}(u)\right) = 0$$

$$= span\{S_i N\}_{i=1}^n \oplus span\{SN_i\}_{i=1}^n$$

or explicitly by:

$$\left(2\delta_{ij} + S(u) l_{ik}(u) \hat{g}^{kj}(u)\right) = 0, \quad i = 1, 2, \ldots, n \tag{12}$$

Recall the following results for the hyp ersurface $\hat{M} = \hat{X}(U)$ in R^{n+1} [4].

Proposition 1. Suppose that $\hat{M} = \hat{X}(U)$ is totally umbilical, ($\hat{k}_p$ is constant $\hat{k}$). Thus we have:

1) If $\hat{k} \neq 0$, then $\hat{M}$ is a part of a hypersphere.

2) If $\hat{k} = 0$, then $\hat{M}$ is a part of a hyperplane.

Proposition 2. For the hypersurface $\hat{M} = \hat{X}(U)$ in R^{n+1} .The following are equivalent:

1) $\hat{M}$ is totally umbilic with $\hat{k} = 0$.

2) The Gauss map is a constant map.

3) $\hat{M}$ is a part of a hyperplane.

$$\hat{N}_1(u) \times \hat{N}_2(u) \times \ldots \times \hat{N}_n(u) = \underline{0}$$

2. Singularities of Gauss Map of pedal hypersurface

The Gauss map is singular at q ∈ $\hat{M}$ when $\hat{K} = 0$, i. e., on the parabolic set given by equation (12).

From the relation:

$$\hat{N}_1(u) \times \hat{N}_2(u) \times \ldots \times \hat{N}_n(u) = \hat{K}(u)\left(\hat{X}_1(u) \times \hat{X}_2(u) \times \ldots \times \hat{X}_n(u)\right) \tag{13}$$

From this definition it follows that the Gauss map is singular when

i.e, the Jacobian matrix of the normal vector field $\hat{N}(u)$ is singular [5],[6],[7].

Gauss map has a singular point u = u_0 when the rank of the Jacobian matrix of $\hat{N}(u_0)$ less than n (the dim of U), i.e.,

$$rank(D\hat{N}(u_0)) = rank((\hat{N}_1(u_0) \times \hat{N}_2(u_0) \times \dots \times \hat{N}_n(u_0)) < n \quad (14)$$

To study the singular points of Gauss map we make a modification of Gauss map as follows: if the Gauss map has a singular point we make projection of N (u) to the (n+1) hyperplanes projections) and find the set of singular points of it (discriminant set).Thus we have

$$\text{Singular set } (s) = \det\left(\frac{\partial(\hat{N}^1(u), \hat{N}^2(u), \dots, \hat{\hat{N}}^k(u), \dots, \hat{N}^{n+1}(u))}{\partial(u_1, u_2, \dots, u_n)}\right) = 0 \quad (15)$$

where the position $\hat{\hat{N}}^k(u)$ indicates that the component $\hat{N}^k(u)$ is missing, The singular set give us a new hypersurface with dimension n which can be written, using the Mong form, as the following:

$$u_n = Z(u_1, u_2, \dots, u_{n-1}) \quad (16)$$

The parametrization (16) define a hypersurface (singular set) contains or may not contains singular points inheritance from the main hypersurface M . For study the graph of singular set under the Gauss map, consider the form

$$\hat{N}(s) = \hat{N}(u_1, u_2, \dots, u_n) = \hat{N}(u_1, u_2, \dots, Z(u_1, u_2, \dots, u_{n-1})) \quad (17)$$

Definition 1. [8] A map f: $R^n \to R^m$ has a singularity of type S_k at the point u_0 if the rank of f at u_0 is min(m,n)-k, the number k is called the deficiency of the singularity, if k = 0 then u_0 is regular point.

Definition 2. [9] The level set attached to the hypersurface M is defined as the following: let $u_n = Z(u_1, u2, \dots, u_{n-1}) = c$, c is constant, if c = 0 that given the level set $V_0 = \{(u_1, u_2, \dots, u_{n-1}): u_n = 0\}$ and the other level sets are

$$V_c=\{(u_1, u_2, \dots, u_{n-1}): u_n = 0\}, \quad c \neq 0$$

Another version of the definition of level sets is contours as given in the following

Definition 3. We say the point p on a surface M with a parametric representation is a contour point if and only if N.pc=0

Where N is the normal vector field on the surface M and c is the view point. The contour line or contour, for short, of a surface is the set of all its contour points.

The determination of the contour line of a surface in the general case involves a numerical method to find the zeros of a real-valued function of n real variables in a domain $(u^1, u^2, \dots, u^n) \in U$. An algorithm and its implementation can be found in [10].

3. Application

As an application, we consider a hypersurface $M \in R^4$, i.e., we try to study the singularities of Gauss map of pedal hypersurface $\hat{M}$ to the hypersurface M given by:

$$M: \quad x_4 = x_1^2 + x_2^2 - x_3^2 \quad (18)$$

This hyp ersurface can b e given by the regular parametrization

$$M: X(u, v, w) = \{u, v, w, f(u, v, w)\}, \quad (19)$$

$$f = u^2 + v^2 - w^2, (u, v, w) \in U \subset R^3$$

The normal vector field on the hypersurface (19) is given as:

$$N(u, v, w) = \frac{1}{\sqrt{g}}\{-2u, -2v, 2w, 1\} \quad (20)$$

Where $g = 1 + 4u^2 + 4v^2 + 4w^2 \neq 0$

The support function S on the hypersurface M is given by:

$$S(u, v, w) = \frac{-f}{\sqrt{g}} \quad (21)$$

Thus the pedal hypersurface $\hat{M}$ attached to the given hypersurface M is defined by (from (20),(21)):

$$\hat{M}: \quad \hat{X}(u, v, w) = \frac{f}{g}\{2u, 2v, -2w, 1\} \quad (22)$$

The normal vector field $\hat{N}$ on a hypersurface $\hat{M}$ can be obtained (from (22)) as in the form:

$$\hat{N}(u, v, w) = \frac{-8f^2}{g^4}\{u(1 + 8w^2), v(1 + 8w^2), w(1 + 8u^2 + 8v^2), f(3 + 4u^2 + 4v^2 + 4w^2)\} \quad (23)$$

The Jacobian matrix (derivative) of $\hat{N}$ (u, v, w) can be written in the following form:

$$D\hat{N}(u, v, w) = \begin{pmatrix} \hat{N}_u^{(1)} & \hat{N}_v^{(1)} & \hat{N}_w^{(1)} \\ \hat{N}_u^{(2)} & \hat{N}_v^{(2)} & \hat{N}_w^{(2)} \\ \hat{N}_u^{(3)} & \hat{N}_v^{(3)} & \hat{N}_w^{(3)} \\ \hat{N}_u^{(4)} & \hat{N}_v^{(4)} & \hat{N}_w^{(4)} \end{pmatrix} \quad (24)$$

The factors in the matrix of equation (24) are calculated.

The rank of D $\hat{N}$ (u, v, w) at (0, 0, 0) is equal to zero, so $\hat{N}$ (u, v, w) is singular at (0, 0, 0). Thus we have the following:

Lemma 1. The Gauss map of $\hat{M}$ has a singularity of type S_3 at the origin point.

To study the singularities of Gauss map of pedal hypersurface, we use the orthogonal projections on the hyperplanes $x_i = 0$; i = 1, 2, 3, 4. Thus we have four surfaces are denoted by σ_i respectively, which are given explicitly by as (Fig(1),(2)) :

(1) σ_4 : $x_4 = 0,\ x_1^2 + x_2^2 - x_3^2 = 0$ (cone).

(2) σ_3 : $x_3 = 0,\ x_4 - x_1^2 - x_2^2 = 0$ (paraboloid of revolution).

(3) σ_2 : $x_2 = 0,\ x_4 - x_1^2 + x_3^2 = 0$ (hyperbolic paraboloid).

(4) σ_1 : $x_1 = 0,\ x_4 - x_2^2 + x_3^2 = 0$ (hyperbolic paraboloid).

Thus, we have 4 parabolic sets S_I, S_{II}, S_{III}, S_{IV} corresponding to the hyperplanes X_i respectively. Using the modified normal vector field $\hat{N}_{mod}$ (u, v, w) for each parabolic set as in the following:

I- For the surface σ_4, we have a modified normal vector field, as in the form:

$$S_I:\quad \hat{N}_{mod}(u,v,w) = \frac{-8f^2}{g^4}\{u\left(1+8w^2\right), v\left(1+8w^2\right), w\left(1+8u^2+8v^2\right)\} \tag{25}$$

The singular (discriminant or parabolic set) set in this case is given from:

$$S_I = \det(D\hat{N}_{mod}(u,v,w)) = -\frac{512}{g^{13}}(f)^6\left(1+8w^2\right)\left(5+28v^2+28w^2+4\left(7u^2-24u^4-48u^2v^2-24v^4+32\left(u^2+v^2\right)\left(3+2u^2+2v^2\right)w^2+8\left(-3+8u^2+8v^2\right)w^4\right)\right)=0 \tag{26}$$

Since $1+8w^2 \neq 0$ ($w \in R$), thus the singular set S_I consists of 2 types of singularity as in the following (fig. 3):

$$S_{I_1}:\quad f = 0,$$

$$S_{I_2}: (5+28v^2+28w^2+4\left(7u^2-24u^4-48u^2v^2-24v^4+32\left(u^2+v^2\right)\left(3+2u^2+2v^2\right)w^2+8\left(-3+8u^2+8v^2\right)w^4\right))=0 \tag{27}$$

Then the parabolic surfaces M_{I_1} corresponding to S_{I_1}, is given by the parametrization:

$$M_{I_1}:\quad (u, v, \sqrt{u^2+v^2}) \quad \text{and} \quad \hat{N}_{mod}(M_{I_1}) = (0,0,0) \tag{28}$$

For the 2nd type S_{I_2} we have four roots, w_i(i =1, 2, 3, 4) as functions in u and v , so we have four pranches $S_{I_{2i}}$, i =1, 2, 3, 4 and their corresponding parabolic surfaces $M_{I_{2i}}$ are given as:

$$M_{I_{2i}}:\quad (u, v, w_i(u,v))$$

The surfaces S_{I_2} under the modified normal vector field $\hat{N}_{mod}$ (u, v, w) and their con-tours are shown by the fig. ((4),(5)).

II- For the surface σ_3, we have a modified normal vector field, as in the form:

$$S_{II}:\quad \hat{N}_{mod}(u,v,w) = \frac{-8f^2}{g^4}\{u\left(1+8w^2\right), v\left(1+8w^2\right), f\left(3+4u^2+4v^2+4w^2\right)\} \tag{29}$$

The singular (discriminant or parabolic set) set in this case is given from:

$$S_{II} = \det(D\hat{N}(u,v,w)) = -\frac{1024w}{g^{13}}f^6\left(1+8w^2\right)\left(-9-12v^2-76w^2+4\left(-3u^2+56u^4+112u^2v^2+56v^4+32\left(u^2+v^2\left(-1+6u^2+6v^2\right)w^2+8\left(-1+24u^2+24v^2\right)w^4\right)\right)\right)=0 \tag{30}$$

Since $1+8w^2 \neq 0$ ($w \in R$), thus the singular set S_{II} consists of 3 types of singularity as in the following (fig. 6):

$$S_{II_0}:\quad w = 0,\quad S_{II_1}:\quad f = 0,$$

$$S_{II_2}: (-9-12v^2-76w^2+4\left(-3u^2+56u^4+112u^2v^2+56v^4+32\left(u^2+v^2\right)\left(-1+6u^2+6v^2\right)w^2+8\left(-1+24u^2+24v^2\right)w^4=0 \tag{31}$$

Then the parabolic surfaces M_{II_1} corresponding to S_{II_1}, is given by the parametrization:

$$M_{II_1}:\quad (u, v, \sqrt{u^2+v^2}) \quad \text{and} \quad \hat{N}_{mod}(M_{II_1}) = (0,0,0) \tag{32}$$

For the 3rd type S_{II_2} we have four roots, w_i(i =1, 2, 3, 4) as functions in u and v , so we have four pranches $S_{II_{2i}}$, i =1, 2, 3, 4 and their corresponding parabolic surfaces $M_{II_{2i}}$ are given as:

$$M_{II_{2i}}:\quad (u, v, w_i(u,v))$$

The surfaces S_{II_0} and $S_{II_{2i}}$ under the modified normal vector field $\hat{N}_{mod}$ (u, v, w) and their contours are shown by the fig. (7), (8),(9).

III- For the surface σ_2, we have a modified normal vector field, as in the form:

$$S_{III}:\quad \hat{N}_{mod}(u,v,w) = \frac{-8f^2}{g^4}\{u\left(1+8w^2\right), w\left(1+8u^2+8v^2\right), f\left(3+4u^2+4v^2+4w^2\right)\} \tag{33}$$

Similarly as in the case I, II, $1+8w^2 \neq 0$ ($w \in R$), thus the singular set S_{III} consists of 3 types of singularity as in the following (fig. 10)

$$S_{III_0}:\quad v = 0,\quad S_{III_1}:\quad f = 0,$$

$$S_{III_2}: (-9-76v^2-4\left(8u^4+8v^4+u^2\left(19+16v^2\right)\right)-12w^2+128\left(u^2+v^2\right)\left(-1+6u^2+6v^2\right)w^2+32\left(7+24u^2+24v^2\right)w^4)=0 \tag{34}$$

Then the parabolic surfaces M_{III_1} corresponding to S_{III_1}, is given by the parametrization:

$$M_{III_1}:\quad (u, v, \sqrt{u^2+v^2}) \quad \text{and} \quad \hat{N}_{mod}(M_{III_1}) = (0,0,0) \tag{35}$$

For the 3rd type S_{III_2} we have four roots, v_i(i =1, 2, 3, 4) as functions in u and w , so we have four pranches $S_{III_{2i}}$, i =1, 2, 3, 4 and their corresponding parabolic surfaces $M_{III_{2i}}$ are given as:

$$M_{III_{2i}}:\quad (u, v, v_i(u,w))$$

The surfaces S_{III_0} and $S_{III_{2i}}$ under the modified normal vector field $\hat{N}_{\text{mod}}$ (u, v, w) and their contours are shown by the fig. (11), (12),(13).

IV- For the surface σ_1, we have a modified normal vector field, as in the form:

$$S_{IV}: \quad \hat{N}_{mod}(u,v,w) = \frac{-8f^2}{g^4}\{v\left(1+8w^2\right), w\left(1+8u^2+8v^2\right), f\left(3+4u^2+4v^2+4w^2\right)\} \quad (36)$$

Similarly the singular set of this case is given from:

$$S_{IV} = \det(D\hat{N}(u,v,w)) = uf^6 \left(-9-76v^2-4\left(8u^4+8v^4+u^2\left(19+16v^2\right)\right)-12w^2+128\right.$$
$$\left.\left(u^2+v^2\left(-1+6u^2+6v^2\right)w^2+32\left(7+24u^2+24v^2\right)w^4\right)=0 \quad (37)$$

Thus the singular set S_{IV} consists of 3 types S_{IV_0}, S_{IV_1}, S_{IV_2} corresponding to (fig. 10):

$$u=0, \qquad f=0,$$
$$\left(-9-76v^2-4\left(8u^4+8v^4+u^2\left(19+16v^2\right)\right)-12w^2+128\right.$$
$$\left.\left(u^2+v^2\right)\left(-1+6u^2+6v^2\right)w^2+32\left(7+24u^2+24v^2\right)w^4\right)=0 \quad (38)$$

Respectively.

Then the parabolic surfaces corresponding to S_{IV_1} is coincident with:

$$M_{IV1}: \quad (u,v,\sqrt{u^2+v^2}) \quad \text{and} \quad \hat{N}_{mod}(M_{IV1}) = (0,0,0) \quad (39)$$

For the 3rd type S_{IV_2} we have four roots, u_i(i =1, 2, 3, 4) as functions in v and w , so we have four pranches $S_{IV_{2i}}$, i =1, 2, 3, 4 and their corresponding parabolic surfaces $M_{IV_{2i}}$ are given as:

$$M_{IV_{2i}}: \quad (u, v, v_i(u,w))$$

The surfaces S_{IV_0} and $S_{IV_{2i}}$ under the modified normal vector field $\hat{N}_{\text{mod}}$ (u, v, w) and their contours are shown by the fig. (11), (12),(13).

4. Conclusion

From (28), (32), (35) and (39), one can see that there exist a common intersection between the singular sets and the four projections where $M_{I_1} \equiv M_{II_1} \equiv M_{III_1} \equiv M_{IV_1}$ and $M_{I_2} \equiv M_{II_2} \equiv M_{III_2} \equiv M_{IV_2}$. The analytical solutions for the singularities of the Gauss maps and their contours on the pedal hypersurface are geometrically interpreted as show in fig. ((4), (5), (7), (8), (9), (11),(12), (13))(left), Also, the contour problem and the problem to find lines of intersection of surfaces and planes has been solved in the general case as shown in fig. ((4), (5), (7), (8), (9), (11), (12), (13))(right).

Figures

Figure 1: σ_4 $(left), \sigma_3$ $(right)$

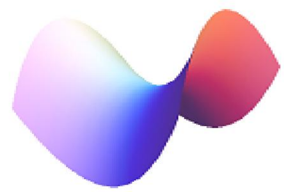

Figure 2: σ_2 or σ_1

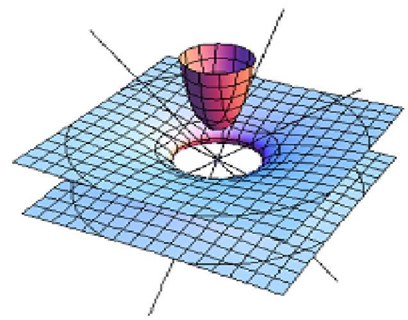

Figure 3: The singular set S_I

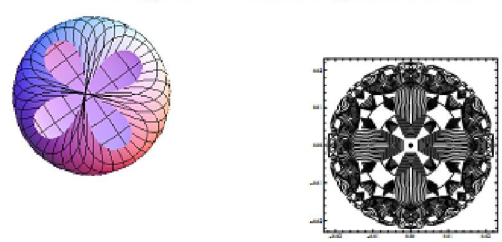

Figure 4: The image of singular $S_{I_{21}} \equiv S_{I_{22}}$ and its contours

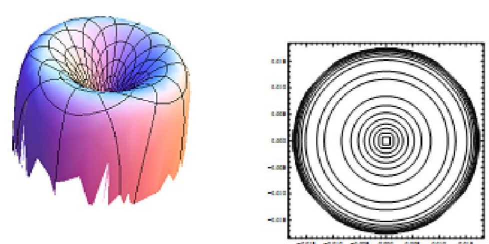

Figure 5: The image of singular $S_{I_{23}} \equiv S_{I_{24}}$ and its contours

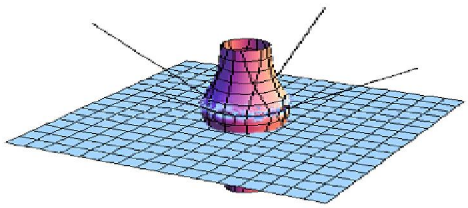

Figure 6: The singular set S_{II}

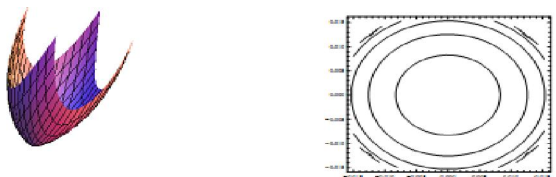

Figure 7: The image of singular S_{II_0} and its contours

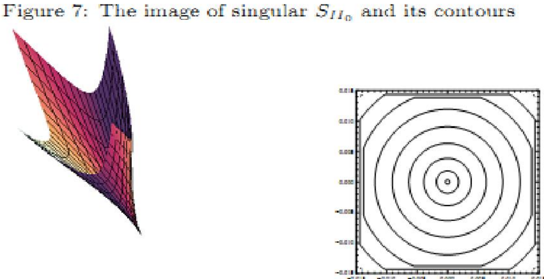

Figure 8: The image of singular $S_{II_{21}} \equiv S_{II_{22}}$ and its contours

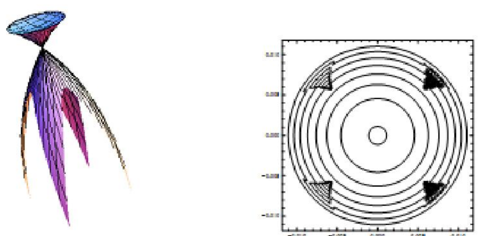

Figure 9: The image of singular $S_{II_{23}} \equiv S_{II_{24}}$ and its contours

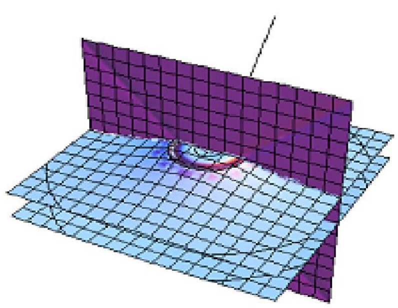

Figure 10: The singular set $S_{III} \equiv S_{IV}$

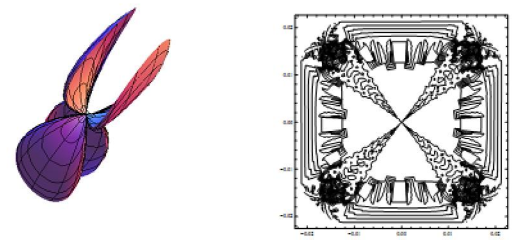

Figure 11: The image of singular set $S_{III_0} \equiv S_{IV_0}$ and its contours

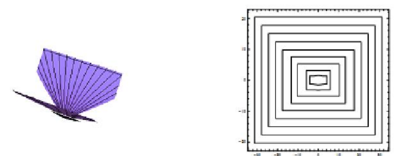

Figure 12: The image of singular set $S_{III_{21}} \equiv S_{III_{22}} \equiv S_{IV_{21}} \equiv S_{IV_{22}}$ and its contours

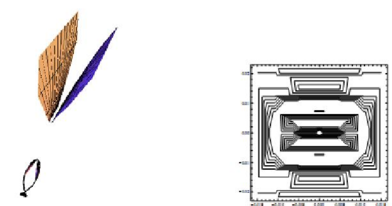

Figure 13: The image of singular set $S_{III_{23}} \equiv S_{III_{24}} \equiv S_{IV_{23}} \equiv S_{IV_{24}}$ and its contours

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