TL-Moments and L-Moments for Order Statistics From Nonidentically Distributed Random Variables with Applications

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Abstract: In this paper, the trimmed L-moments (TL-moments) and L-moments of order statistics from nonidentically distributed random variables will be derived and used to obtain the first four TL-moments and L-moments for some distributions: Erlang Truncated Exponential – Beta type I with three parameters and Burr type II. TL-skewness, L-skewness, TL-kurtosis and L-kurtosis are handled for these distributions. A numerical illustration for the new results will be given.

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1. Introduction

The method of L-moment estimators have recently appeared. (Hosking, 1990) gives estimators for log-normal, gamma and generalized extreme value distributions. L-moment estimators for generalized Rayleigh distribution was introduced by (Kundu and Raqab, 2005). (Karvanen, 2006) applied the method of L-moment estimators to estimate the parameters of polynomial quantile mixture. He introduced the mixture composed of two parametric families; the normal-polynomial quantile and Cauchy-polynomial quantile. The standard method to compute the L-moment estimators is to equate the sample L-moments with the corresponding population L-moments. A population L- moment L_r is defined to be a certain linear function of the expectations of the order statistics $Y_{1:r}, Y_{2:r},..., Y_{r:r}$ in a conceptual random sample of size r from the underlying population. For example, $L_1 = E(Y_{1:1})$, which is the same as the population mean, is defined in terms of a conceptual sample of size $r = 1$, while $L_2 = (1/2)E(Y_{2:2} - Y_{1:2})$, an alternative to the population standard deviation, is defined in terms of a conceptual sample of size $r = 2$. Similarly, the L-moments L_3 and L_4 are alternatives to the un-scaled measures of skewness and kurtosis μ_3 and μ_4 respectively. See (Silito, 1969). Compared to the conventional moments, L-moments have lower sample variances and are more robust against outliers. (Elamir and Seheult, 2003) introduced an extension of L-moments called TL-moments. TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments exist for Cauchy distribution. (Abdul-Moniem, 2007)(Abdul-Moniem, 2009) applied the method of L-moment and TL-moment estimators to estimate the parameters of exponential distribution. The following formula gives the r^{th} TL-moments see (Elamir and Seheult, 2003).

$$L_r^{(t)} = \frac{1}{r}\sum_{k=1}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r+t-k:r+2t}), \qquad (1)$$

Where r and t take the values 1, 2, 3, ... Note that the r^{th} L-moments can be obtained by taking $t = 0$.

The main aim of this paper is to derive TL-moments and L-moments of the order statistics (os) from independent Nonidentically distributed (INID) random variables (rv's). The subject of os for INID rv's and their moments for discrete and continuous distributions are getting much attention in the literature lately Barakat H, Abdelkader Y(2003) Jamjoom A., Al-Saiary Z.(2013) Jamjoom A. A.(2013. This paper is organized as follows: in Section 2, we introduced population TL- moments and L-moment. The population TL-moments and L-moment for the Erlang Truncated Exponential (ETE), Burr type II and Beta type I were presented in Section 3. In Section 4, a numerical illustrate for the new results for Burr type II is given.

2. TL-moments and L-moments for the (INID)

In this section, the population TL-moment and L-momentsof order r for INID will be obtained.

2.1 Population TL-moments

Using formula (1) and the following theorem from(Barakat and Abdelkader, 2003):

Theorem 2.1: Let $X_1, X_2, \cdots, X_n$ be independent nonidentically distributed r.v.s. The k moment of all order statistics, $\mu_{r:n}^{(k)}$,for $1 \le r \le n$ and $k = 1, 2, \ldots$ is given by:

$$\mu_{r:n} = \sum_{i=n-r+1}^{n} (-1)^{i-(n-r+1)} \binom{i-1}{n-r} I_i \,(1) \tag{2}$$

Where:

$$I_i\,(1) = \sum_{1 \le j_1 < j_2 < \cdots j_i \le n} \cdots \sum \int_0^\infty \prod_{m=1}^{i} G_{j_m}(x)\,dx\,,\; i = 1, 2, \cdots, n \tag{3}$$

$G_{j_m}(x) = 1 - F_{j_m}(x)$, with $(j_1, j_2, \cdots, j_n)$ is a permutation of (1, 2, …, n) for which $i_1 \le i_2 < \cdots < i_n$.

$$E(X_{r+t-k\,:r+2t} = \mu_{r+t-k\,:r+2t}^{(k)} = \sum_{i=t+k+1}^{r} (-1)^{i-(t+k+1)} \binom{i-1}{t+k} I_i$$

Where

$$I_i = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r+2t} \cdots \sum \int_0^\infty \prod_{m=1}^{i} [1 - F_{j_m}(x)]\,dx \tag{4}$$

Equation (1) becomes:

$$L_r^{(t)} = \frac{1}{r} \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} \sum_{i=t+k+1}^{r} (-1)^{i-(t+k+1)} \binom{i-1}{t+k} I_i$$

$$= \frac{1}{r} \sum_{k=1}^{r-1} \sum_{i=t+k+1}^{r} (-1)^{i-t-1)} \binom{r-1}{k} \binom{i-1}{t+k} I_i$$

$$= \sum_{i=t+1}^{r+2t} (-1)^{i-t-1)} \left[\frac{1}{r} \sum_{k=0}^{i-1} \binom{r-1}{k} \binom{i-1}{t+k} \right] I_i$$

Let: $\left[\frac{1}{r} \sum_{k=0}^{i-1} \binom{r-1}{k} \binom{i-1}{t+k} \right] = b\,(i; r, t)$

$$\therefore L_r^{(t)} = \sum_{i=t+1}^{r+2t} (-1)^{i-t-1} b\,(i; r, t)\, I_i \tag{5}$$

From equation (4) in (5) we get:

$$\therefore L_r^{(t)} = \sum_{i=t+1}^{r+2t} (-1)^{i-t-1} b\,(i; r, t)$$

$$\times \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r+2t} \cdots \sum \int_0^\infty \prod_{m=1}^{i} [1 - F_{j_m}(x)]\,dx\,, \tag{6}$$

where $r, t = 1, 2, 3, \ldots$.

2.2 Population L-moments

If t = 0 in equation (6) we obtain the L-moments from (INID) as

$$\therefore L_r = \sum_{i=1}^{r} (-1)^{i-1} b\,(i,r) \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r} \cdots \sum \int_0^{\infty} \prod_{m=1}^{i} [1 - F_{j_m}(x)]dx \quad (7)$$

where $r = 1, 2, 3, \ldots$.

3. The population TL-moments and L-moments for some distributions

3.1 The population TL-moments and L-moment for the Erlang truncated Exponential:

The distribution function of Erlang truncated Exponential distribution can be written as (El-Alosey, 2007) & (Mohsin, 2009):

$$F_j(x) = 1 - e^{-\beta x (1 - e^{-\alpha_j})}, \; 0 \le x \le \infty, \; \beta, \alpha_j > 0, j = 1,2,\ldots,n$$

$$L_r^{(t)} = \sum_{i=t+1}^{r+2t} (-1)^{i-t-1} b\,(i;r,t)$$

$$\times \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r+2t} \cdots \sum \frac{1}{\beta [i - \sum_{m=1}^{i} e^{-\alpha_{j_m}}]} \quad (8)$$

$$L_r = \sum_{i=1}^{r} (-1)^{i-1} b\,(i,r) \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r} \cdots \sum \frac{1}{\beta [i - \sum_{m=1}^{i} e^{-\alpha_{j_m}}]} \quad (9)$$

3.2 The population TL-moments and L-moments for Beta type I

The distribution function of Beta type I can be written as (Johnson et al., 1994)

$$F_i(x) = 1 - \left[\frac{\delta - x}{\delta - w}\right]^{p_i}, \; w \le x \le \delta, \; p_i > 0, \quad \delta, w > 0$$

$$L_r^{(t)} = \sum_{i=t+1}^{r+2t} (-1)^{i-t-1} b\,(i;r,t)$$

$$\times \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r+2t} \cdots \sum \frac{\delta^{1+\sum_{m=1}^{i} p_{j_m}}}{(\delta - w)^{\sum_{m=1}^{i} p_{j_m}}} \beta_{\frac{\delta - w}{\delta}}\left(\sum_{m=1}^{i} p_{j_m} + 1, 1\right) \quad (8)$$

3.3 The population TL-moments and L-moments for Burr type II

The distribution function of Burr type II distribution can be written as:

$$F_i(x) = 1 - \left[1 + e^{x}\right]^{-m_i}, \; -\infty \le x \le \infty, m_i > 0, i = 1,2,\cdots,n$$

See (Ragab et al., 1991).

$$L_r^{(t)} = \sum_{i=t+1}^{r+2t} (-1)^{i-t-1} b\,(i;r,t) \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r+2t} \cdots \sum \left[\psi(1) - \psi\left(\sum_{b=1}^{i} m_{j_b}\right)\right] \quad (8)$$

$$L_r = \sum_{i=1}^{r} (-1)^{i-1} b(i,r) \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r} \cdots \sum \left[\psi(1) - \psi\left(\sum_{b=1}^{i} m_{j_b} \right) \right] \quad (9)$$

Table 1. The mean $\mu_{r:n}$ of all order statistics from INID Burr type II random variables.

n \ r	1	2	3	4	5
2	-2.08333	-0.716667			
3	-1.50000	0.105922	2.16667		
4	-2.45	-1.17106	-0.248413	0.975397	
5	-2.82897	-1.62612	-0.854759	-3069.11	1.02362

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