APPROXIMATION IN CHAOTIC SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS BY EULER METHOD AND CONTROLLING BY ARRAY METHOD

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Abstract. In this article, we study an approximation of a system of differential equations when it has a noise. We use the Taylor method and we model the organization of such systems. In a system of differential equations, we set a scalar multiplication with a function and we saw that this system can be in chaotic mode. We used a method to omit the noises and chaos in this system.

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1. Introduction

In many natural and real phenomenon, a system of differential equations can be seen. These systems may have different behavior in different qualifications [4]. One of these cases is chaotic qualification. A system in chaotic qualification may have many stable and/or unstable points. This system in bounded space have many iterates around several strange attractors. A system of differential equations can be chaotic when we add a function to it. H. Poincare proved that we cannot solve three mass models, about 150 years ago. We can solve this problem with Poincare section [1]. In this paper we want to study these systems and controlling the dynamic of those systems in chaotic qualification. We use control parameters for a system of differential equations when we add a matrix function to it. We introduced Chaos in Section 2, and explain about a system of differential equations in Section 3 and set several term and definition about this. Some numerical examples are given in Section 4.

2. Chaos

A dictionary definition of chaos is a disordered state of a collection; a confused mixture. This is an accurate description of dynamical systems theory today or of any other lively field of research. (Morris W. Hirsch).

When a system in nature is mathematically modeled, we find that their graphical representations are not straight light lines and the system behavior is not so easy to predict. After researches on complex systems, now we know that noise is actually important information about the experiments. When noise is inserted in to the result graph, the graph no longer appear as straight line, neither its point are predictable. Once, this noise was referred to as the chaos in an experiment. For chaos applications we can mention, much like physics, chaos theory provides a foundation for the study of all other scientific disciplines. It is actually a toolbox of methods for incorporating non-linear dynamics for the study of science.

3. A System of Differential Equations

Let $f_i:\mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be a function, for $i = 1,2,\dots,n$. Also let ' be the derivative operator $\frac{\partial}{\partial t}$.

For $t \in [a,b]$, consider the following system of differential equations:

$$\begin{cases} x_1' = f_1(x_1,x_2,\dots,x_n,t) \\ x_2' = f_2(x_1,x_2,\dots,x_n,t) \\ \quad\vdots \\ x_n' = f_n(x_1,x_2,\dots,x_n,t) \end{cases} \tag{3.1}$$

where for $i = 1,2,\ldots,n$. $x_i \in [a,b] \longrightarrow \mathbb{R}$ is a function of t.

Definition 3.1. *Let f be a scalar function or an operator, also* $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)$ *be a vector. We define* $f(\Phi)$ *as follows*

$$f(\Phi) = (f(\phi_1), f(\phi_2), \ldots, f(\phi_n))^T \tag{3.2}$$

We can rewrite the system of differential equations (3.1), as follows:

$$X' = F(X,t), \quad t \in [a,b] \tag{3.3}$$

where $F(X,t) = (f_1(X,t), f_2(X,t), \ldots, f_n(X,t))^T$. The function F($X$, t) is satisfied the Lipchitz condition with Lipchitz constant L if for any two points (X, t) and (Y, t) we have:

$$\|F(X,t) - F(Y,t)\| \leq L\|X - Y\| \tag{3.4}$$

Definition 3.2. *Let* $V = (v_1, v_2, \ldots, v_n)^T$ *and* $W = (w_1, w_2, \ldots, w_n)^T$ *be two vectors. We define a direct product,* $V \odot W$ *as follows:*

$$V \odot W = (v_1 w_1, v_2 w_2, \ldots, v_n w_n) \tag{3.5}$$

for the purpose of solving (3.1) numerically by using Euler method we have:

$$X^{(k+1)} = X^{(k)} + hF(X^{(k)}, t) \tag{3.6}$$

where $H = (h_1, h_2, \ldots, h_n)$ and $\odot$ is a direct product.

Theorem 3.3. *Let* a_k *be a sequence of real numbers such that* $a_0 > 0$ *and there exist two constant* α *and* β*, such that for any k we have* [2]

$$a_k \leq (1+a)a_{k-1} + \beta \tag{3.7}$$

Therefore for any k we have:

$$a_k \leq e^{ka}\left(\frac{\beta}{\alpha} + a_0\right) - \frac{\beta}{\alpha} \tag{3.8}$$

Theorem 3.4. *Let* $X^{(k)}$ *be a vector obtained from Euler method in step k, and for* $i = 1,2,\ldots,n$ *we have* $|X''| \leq M$, *where* $M = (M_1, M_2, \ldots, M_n)$. *Suppose that* $\bar{X}^{(k)}$ *be the exact vector in step k. Thus we have*

$$\left|\hat{X}^{(k)} - X^{(k)}\right| \leq \frac{1}{2L} hM \odot \left(e^{L\left(X^{(k)} - X^{(0)}\right)}\right) \tag{3.9}$$

Proof. when know that

$$x' = f_i(x_1, x_2, \ldots, x_n, t)$$

From Euler method we have

$$X^{(k)} = X^{(k-1)} + hF(X^{(k-1)}, t)$$

Let $\hat{X}^{(k)}$ be the exact vector in step $\hat{X}^{(k)} = X(t_k)$, thus

$$X^{(k)} = X^{(k-1)} + hF\left(X^{(k-1)}, t\right) + \frac{1}{2!}h^2 F'(X^{(k-1)}, \eta_t)$$

where $\eta_t \in (t_k, t_{k+1})$.Therefore

$$\left|\hat{X}^{(k)} - X^{(k)}\right| \leq \left|\hat{X}^{(k-1)} - X^{(k-1)}\right| + Lh\left|\hat{X}^{(k-1)} - X^{(k-1)}\right| + \frac{1}{2!}h^2 \odot F'(X^{(k-1)}, \eta_t)$$

Also we know that $F'\left(X^{(k-1)}, \eta_t\right) \leq M$ so we have:

$$\left|\hat{X}^{(k)} - X^{(k)}\right| \leq \left|\hat{X}^{(k-1)} - X^{(k-1)}\right| + Lh\left|\hat{X}^{(k-1)} - X^{(k-1)}\right| + \frac{1}{2!}h^2 M$$

From Lemma 3 we have

$$\left|\hat{X}^{(k)} - X^{(k)}\right| \leq \frac{1}{2L} hM \odot \left(e^{L\left(X^{(k)} - X^{(0)}\right)}\right)$$

Corollary 3.1. *From Theorem 4 we have:*

$$\left\|\hat{X}^{(k)} - X^{(k)}\right\| \leq \frac{|h|\|M\|}{2L}\left(e^{L\left\|X^{(k)} - X^{(0)}\right\|}\right) \tag{3.10}$$

Now let we add a function to this system of differential equations. In lots of natural and real phenomenon, we need to do such work. For example when an airplane goes through a storm, or when a robot faces a snag or any one has a epilepsy. Another example is the case when we have an artificial moon between the real moon and earth. All of this phenomenon and other example are satisfied by this term.

Let $G(X,t) = (g_1(X,t), \dots, g_n(X,t))^T$ be a function. Now consider the new system of differential equations:

$$X' = \Phi(X,t) = F(X,t) + G(X,t)$$

The functions $F(X, t)$ and $G(X, t)$ are satisfied in Lipchitz condition with Lipchitz constants L and $\bar{L}$, res (3.11)

So for $\Phi(X,t)$ we have:

$$\|\Phi(X,t) - \Phi(Y,t)\| \le (L + \bar{L})\|X - Y\| \tag{3.12}$$

Lemma 3.1 *Let $X^{(k)}$ be a vector obtained from Euler method in step k, and for $t \in [a,b]$, we have $|F'| \le M$, $|G'| \le \bar{M}$ where $M = (M_1, \cdots, M_n)$ and $\bar{M} = (\bar{M}_1, \dots, \bar{M}_n)$. Suppose that $\hat{X}^{(k)}$be the exact vector in step k. Thus we have:*

$$\left|\hat{X}^{(k)} - X^{(k)}\right| \le \frac{1}{2(L+\bar{L})} h(M + \bar{M}) \odot \left(e^{(L+\bar{L})(X^{(k)} - X^{(0)})}\right) \tag{3.13}$$

Proof. It is a corollary of Theorem 4.

Corollary 3.2. *From Lemma 3.1 we have:*

$$\left\|\hat{X}^{(k)} - X^{(k)}\right\| \le \frac{|h|(\|M\|+\|\bar{M}\|)}{2(L+\bar{L})}\left(e^{(L+\bar{L})\|X^{(k)}-X^{(0)}\|}\right) \tag{3.14}$$

This Corollary shows that the maximum error is affected by the added function. Therefore the system can be unstable if $g_i(X)$ is unbounded or has a large bound, for some of indexes i.

We use a parameter ε for $G(X; t)$, to control the behavior of the system. So we consider a system as:

$$X' = \Phi(X,t) = F(X,t) + \varepsilon G(X,t) \tag{3.15}$$

The functions $F(X; t)$ and $G(X; t)$ are satisfied in Lipchitz condition with Lipchitz constants L and $\bar{L}$, respectively. So for $\Phi(X,t)$ we have:

$$\|\Phi(X,t) - \Phi(Y,t)\| \le (L + \varepsilon\bar{L})\|X - Y\| \tag{3.16}$$

Theorem 3.5. *By assumptions of Lemma 3.1 and Corollary 2, we have:*

$$\left\|\hat{X}^{(k)} - X^{(k)}\right\| \le \frac{|h|(\|M\|+|\varepsilon|\|\bar{M}\|)}{L+\bar{\varepsilon}L}\left(e^{(L+\varepsilon L)\|X^{(k)}-X^{(0)}\|}\right) \tag{3.17}$$

If we use vector ϵ as a control array parameter, we will have:

$$X' = \Phi(X,t) = F(X,t) + \epsilon \odot G(X,t)$$

Theorem 3.6. *By assumptions of Lemma 3.1 and Corollary 2, we have:*

$$\left\|\hat{X}^{(k)} - X^{(k)}\right\| \le \frac{|h|(\|M\|+\|\epsilon\|\|\bar{M}\|)}{L+\bar{\varepsilon}L}\left(e^{(L+\varepsilon L)\|X^{(k)}-X^{(0)}\|}\right) \tag{3.17}$$

4. Numerical exampel

Example 1

Consider the following system of differential equations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -ax + ay \\ cx - y - xz \\ xy - bz \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.1}$$

We surveyed this system with Euler method. Let ε varies from 0 to 14. In the following figure the values of $\|X\|$ are shown for the last 100 iterations, when we have 1000 iterations. In this case our parameters are $a = 4, b = 1$. (Figure 1)

Now let $\varepsilon = 14$, the answer of above system in a 3-dimensional space is shown in the following figure for all 2000 iterations. (Figure 2)

Example 2

Consider the following system of differential equations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x(x+1)(1-ax)-y \\ bx \end{pmatrix} - \epsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.2}$$

There we have $\epsilon = (\varepsilon_1, \varepsilon_2)$. We surveyed this system with Euler method. Let $\varepsilon_1 = \varepsilon_2$ and ε_1 varies from -5 to 5. In the following figure the values of $\|X\|$ are shown for the last 100 iterations, when we have 1000 iterations. In this case our parameters are $a = 4$, $b = 1$. (Figure 3)

Now let $\varepsilon_1 = 14$ and $\varepsilon_1 = 0.1$. The answer of above system in a 2-dimensional space is shown in the following figure for all 2000 iterations. (Figure 4)

We surveyed this system with Euler method. Let ε_1 varies from -5 to 5. In the following figure the values of $\|X\|$ are shown for the last 100 iterations, when we have 1000 iterations. In this case our parameters are $a = 4$, $b = 1$ and $\epsilon = (\varepsilon_1, 0.1\varepsilon_1)$. (Figure 5)

As you see the system affects is better than the systems with immovable parameter of ε.

This system will be dynamic when ε_1 is more than about 1.5. Now let $\varepsilon_1 = 0.1$, the answer of above system in a 2-dimensional space is shown in the following figure for all 2000 iterations. (Figure 6)

Example 3

Consider the following system of differential equations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -y-x \\ x+by \\ 0.2+xz \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \tag{4.3}$$

We surveyed this system with Euler method. Let ε varies from -7.7 to -4.9. In the following figure the values of $\|X\|$ are shown for the last 100 iterations, when we have 1000 iterations. In this case our parameters are $a = 4, b = 1$. (Figure 7)

Now let $\varepsilon = -13$, the answer of above system in a 2-dimensional space and 3-dimensional space is shown in the following figure for all 2000 iterations. (Figure 8)

let $\varepsilon = -13$ In the following figure we map the points (x_k, x_{k+1}) for 2000 iterations. (Figure 9)

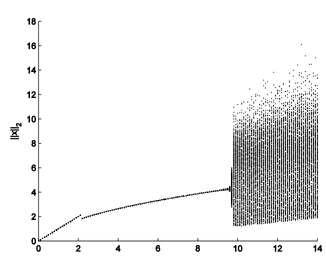

Figure 1: $\|X\|$ for the last 100 iterations when $a = 4, b = 1$

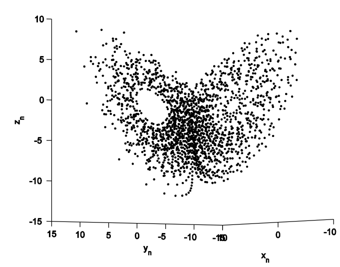

Figure 2: $\|X\|$ in 3-dimensional space all 2000 iterations when $\varepsilon = 14$

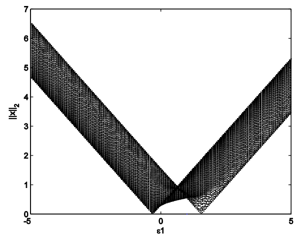

Figure 3: $||X||$ for the last 100 iterations when $a = 4, b = 1$

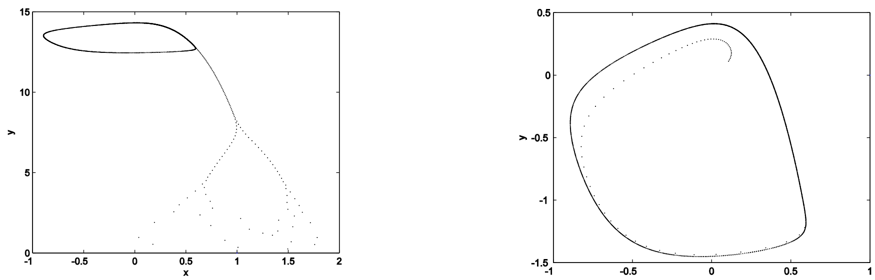

Figure 4: $||X||$ in 3-dimensional space all 2000 iterations when $\varepsilon = 14$ and $\varepsilon_1 = 0.1$

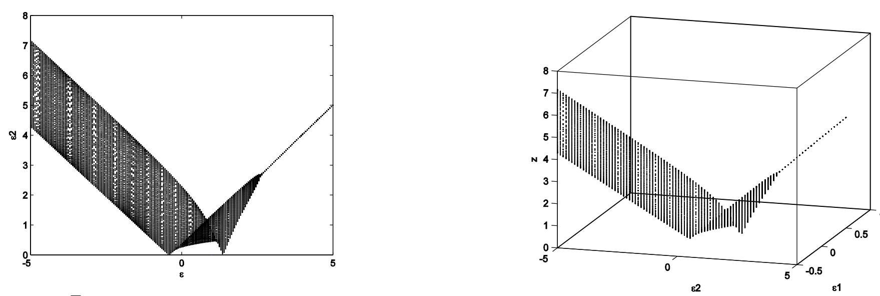

Figure 5: $||X||$ for the last 100 iterations when $a = 4, b = 1$ and $\epsilon = (\varepsilon_1, 0.1\varepsilon_1)$

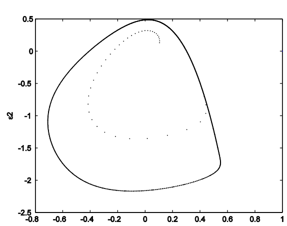

Figure 6: $||X||$ in 3-dimensional space all 2000 iterations when $\varepsilon_1 = 0.1$

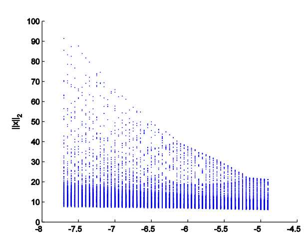

Figure 7: $||X||$ for the last 100 iterations when $a = 4, b = 1$

Figure 8: $\|X\|$ in 2-dimensional and 3-dimensional space all 2000 iterations when $\varepsilon = -13$

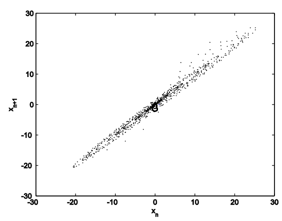

Figure 9: The points (x_k, x_{k+1}) for 2000 iterations when $\varepsilon = -13$

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