Fuzzy Fractional Initial Value Problems

Abdorreza Panahi [1,*] and Azam Noorafkan Zanjani [1]

[1.] Department of Mathematics, Islamic Azad University, Saveh Branch, Saveh, Iran.
Panahi53@gmail.com, Apanahi@iau-saveh.ac.ir

Abstract: In this paper we define fuzzy fractional derivative in Caputo sense. Then using Adomian decomposition method we propose a method for computing approximations of solution of fuzzy fractional initial value problems.
[Abdorreza Panahi, Azam Noorafkan Zanjani. Fuzzy Fractional Initial Value Problems. Journal of American Science 2011;7(4):427-431]. (ISSN: 1545-1003). http://www.americanscience.org.

Key words: Fuzzy initial value problems, Caputo fractional derivative, Adomian decomposition method

1. Introduction

Fractional differential equations with fuzzy initial value will form fuzzy fractional initial value problems. In recent years, fractional differential equations have found applications in many problems in physics and engineering [15], [16]. Benchohra and Darwish [8] introduced an existence and uniqueness theorem for fuzzy integral equation of fractional order and under some assumptions gave a fuzzy successive iterations which were proved to be uniformly convergent to the unique solution of fuzzy fractional integral equation. In the present paper we will use crisp successive iterations. Differential equations which arise in real-word physical problems are often too complicated to solve exactly. We propose a method for computing approximations of solution of a fuzzy fractinal initial value problem using Adomian decomposition method. The Adomian decomposition method has been shown [18] to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate crisp solutions in crisp problems. Here we use the advantage of this method to find an approximate solution for the fuzzy fractional initial value problem with fractional derivative in Caputo sence.

2. Notations and preliminaries

First we recall some basic definitions concerning fuzzy numbers. Let E denote the class of fuzzy sets on the real line.

Definition 2.1. We write $A(x)$, a number in $[0,1]$, for the membership function of A evaluated at x. An α-cut of A written A_α is defined as $\{x \mid A(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. We separately define A_0 as the closure of the union of all the A_α, $0 < \alpha \leq 1$.

The parametric form of a fuzzy number can be defined as follows. According to the representation theorem for fuzzy numbers or intervals [10], we use $\alpha-$cut setting to define a fuzzy number or interval.

Definition 2.2 [10]. A fuzzy number (or interval) u is completely determined by any pair $u = (u^-, u^+)$ of functions $u^\pm : [0,1] \to R$, defining the end-points of the α-cuts, satisfying the three conditions:

(i) $u^- : \alpha \to u_\alpha^- \in R$ is a bounded monotonic increasing (non-decreasing) left-continuous function $\forall \alpha \in (0,1]$ and right-continuous for $\alpha = 0$;

(ii) $u^+ : \alpha \to u_\alpha^+ \in R$ is a bounded monotonic decreasing (non-increasing) left-continuous function $\forall \alpha \in (0,1]$ and right continuous for $\alpha = 0$;

(iii) $u_\alpha^- \leq u_\alpha^+ \ \forall \alpha \in [0,1]$.

If $u_1^- < u_1^+$ we have a fuzzy interval and if $u_1^- = u_1^+$ we have a fuzzy number; for simplicity we refer to fuzzy numbers as intervals.
We will then consider fuzzy numbers of normal and upper semicontinuous form also we assume that the support $[u_0^-, u_0^+]$ of u is compact (closed and bounded). The notation $u_\alpha = [u_\alpha^-, u_\alpha^+]$, $\alpha \in [0,1]$ denotes explicitly the α-cuts of u.

We use a metric in E by the relation

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d(u_\alpha, v_\alpha)$$

where d is the Hausdorff metric [17], for nonempty compact subsets of R.

Definition 2.3 A mapping $F : I \to E$ is strongly measurable if for all α ,($0 < \alpha \leq 1$), $F_\alpha(t)$ is

Lebesgue measurable for any $t \in I$, where $F_\alpha(t) = [F(t)]^\alpha$.

Definition 2.4 A mapping $F: I \to E$ is called levelwise continuous at $t_0 \in I$ if all its α-levels $F_\alpha(t) = [F(t)]^\alpha$ are continuous at $t = t_0$ with respect to the metric d. And F is called integrably bounded if there exist an integrable function h such that $|x| \le h(t)$ for all $x \in [F(t)]^0$.

We define

$F_\alpha(t) = [F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha]$

as α-levels of the mapping $F: I \to E$. A function $f: I \to R$ is said to be a measurable selection of $F_\alpha(t)$ if $f(t)$ is measurable and

$F_1(t,\alpha) \le f(t) \le F_2(t,\alpha)$

for all $t \in I$.

Definition 2.5 Let $F: I \to E$. The integral of F over I is defined levelwise by the equation $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt = \{\int_I f(t)dt \mid f: I \to R$ is a measurable selection for $F_\alpha\}$,

for all $0 < \alpha \le 1$.

Theorem 2.1 If $F: I \to E$ is strongly measurable and integrably bounded, then F is integrable.

Definition 2.6 [16]. Reimann-Liouville's fractional derivative and fractional integral of order r ($0 < r < 1$) for $u(t): R \to R$ are defined as

$$D^r u = \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} u(s) ds$$

and

$$I^r u(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} u(s) ds. \quad (1)$$

Definition 2.7 Caputo fractional derivative of order r ($0 < r < 1$) for $u(t): R \to R$ is defined as

$$D_c{}^r u = \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \left(\frac{d}{ds} u(s)\right) ds \quad (2)$$

Definition 2.8 A function $y(t), t > 0$ is said to be in the space C_μ, $\mu \in R$, if there exist a real number $p > \mu$ such that $y(t) = x^p y_1(t)$, where $y_1(t) \in C(0,\infty)$, and is said to be in the space C_μ^n if and only if $y^{(n)} \in C_\mu$, $n \in N$.

Lemma 2.1 [11]. Let $n-1 < r \le n, n \in N$ and $y(t) \in C_\mu^n$, $\mu \ge -1$, then

$$I^r D_c{}^r y(t) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{t^k}{k!}$$

To perform the Adomian decomposition method, the fractional initial value problem can be modeled as $Ly(t) + Ny(t) + Ry(t) = 0$, where $L = D_c{}^r$ therefore $L^{-1} = I^r$. Since $L^{-1}Ly = y - ct^{r-1}$ then

$$y = ct^{r-1} + -I^r(Ry) - I^r(Ny). \quad (3)$$

The solution y is represented as an infinite sum

$$y = \sum_{n=0}^{\infty} y_n \quad (4)$$

and the nonlinear term Ny will be decomposed by the infinite series of Adomian polynomials

$$Ny = \sum_{n=0}^{\infty} A_n$$

where the A_n s are obtained by writing

$$z(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n$$

$$N(z(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n$$

therefore, for any $n = 0,1,...$

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N(z(\lambda))\right]_{\lambda=0}.$$

Then substituting (3) in (4) we obtain the following relations

$$\sum_{i=0}^{\infty} y_i = ct^{r-1} - \sum_{i=0}^{\infty} I^r(Ry_i) - \sum_{i=0}^{\infty} I^r(A_i)$$

and we define y_0, y_1, y_2, Λ in a recurrent manner.

$$y_0 = ct^{r-1}$$

$$y_1 = -I^r Ry_0 - I^r A_0$$

$$y_2 = -I^r Ry_1 - I^r A_1$$

The truncated series $\sum_{i=0}^{N} y_i$ could be as approximat solution.

3. Fuzzy initial value problem

Until now, there have been several methods to deal with the fuzzy initial value problem. Consider the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0 \in E. \tag{5}$$

There are many suggestions to define a fuzzy derivative and in consequence, to study Eq. (5), see for instance [1]-[7], [9], [12]-[15], [17].

Definition 3.1 A function $F:[a,b] \to E$ is differentiable at a point $t_0 \in (a,b)$, if there is such an element $F'(t_0) \in E$, that the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) \overset{H}{-} F(t_0)}{h} \quad and \qquad \lim_{h\to 0^+} \frac{F(t_0) \overset{H}{-} F(t_0-h)}{h} \tag{I}$$

or

$$\lim_{h\to 0^-} \frac{F(t_0+h) \overset{H}{-} F(t_0)}{h} \quad and \qquad \lim_{h\to 0^-} \frac{F(t_0) \overset{H}{-} F(t_0-h)}{h} \tag{II}$$

exist and are equal to $F'(t_0)$. Here the limits are taken in the metric space (E, D) and $\overset{H}{-}$ is the Hakuhara difference.

In the next section we will define more general type of differentiability for fuzzy fractional differential equations. Now, for $F:[a,b] \to E$, we easily obtain the following result:

Theorem 3.1 [12]. Let $F:[a,b] \to E$ be differentiable and denote $[F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha)]$. Then

(i) If F is differentiable of type (I), then F_1 and F_2 are differentiable and

$[F'(t)]^\alpha = [F_1'(t,\alpha), F_2'(t,\alpha)]$,

(ii) If F is differentiable of type (II), then F_1 and F_2 are differentiable and

$[F'(t)]^\alpha = [F_2'(t,\alpha), F_1'(t,\alpha)]$.

4. Fuzzy fractional initial value problem

In Eq. (5), we will replace the first derivative of x with the fractional derivative of x in Caputo sence. We use the definition of Caputo's fractional derivative of a crisp function in Eq. (2) to define fuzzy fractional derivative. Let $x:[0,T] \to E$ be a fuzzy function of a crisp variable. For $[x(s)]^\alpha = [x_1(s,\alpha), x_2(s,\alpha)]$, we have $[x'(s)]^\alpha = [x_1'(s,\alpha), x_2'(s,\alpha)]$ since $t - s > 0$ then $D_c^{\ r} x(t)$ can be defined levelwise as

$$[D_c^{\ r} x(t)]^\alpha = [D_c^{\ r} x_1(t,\alpha), D_c^{\ r} x_2(t,\alpha)]$$

$$= [\frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} x_1'(s,\alpha) ds,$$

$$\frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} x_2'(s,\alpha) ds]$$

Consider the following initial value problem with fractional derivative

$$\begin{cases} D_c^{\ r} x(t) = f(x,t), & t \in [t_0, T] \\ x(t_0) = x_0 \in E, & t_0 \in [0,T) \end{cases} \tag{6}$$

In general, it is too difficult to find an exact analytical solution for (6), so we will try to find an approximate analytical solution.

Theorem 4.1 Let $0 < r < 1$ and $x:[a,b] \to E$ be a fuzzy function with $[x(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$.
Then

(i) If x has fractional derivative of type (I), then x_1 and x_2 have fractional derivative and

$[D_c^{\ r} x(t)]^\alpha = [D_c^{\ r} x_1(t,\alpha), D_c^{\ r} x_2(t,\alpha)]$

(ii) If x has fractional derivative of type (II), then x_1 and x_2 have fractional derivative and

$[D_c^{\ r} x(t)]^\alpha = [D_c^{\ r} x_2(t,\alpha), D_c^{\ r} x_1(t,\alpha)]$

Proof. We prove part (i) and the same proof can be used for part (ii). Since $0 \le s \le t$ and

$[x(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$

Then

$[x'(s)]^\alpha = [x_1'(s,\alpha), x_2'(s,\alpha)]$

$[(t-s)^{-r} x'(s)]^\alpha = [(t-s)^{-r} x_1'(s,\alpha), (t-s)^{-r} x_2'(s,\alpha)]$.

Since $0 < r < 1$, then $\Gamma(1-r) > 0$, therefore

$$[\frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r} x'(s)ds]^\alpha =$$

$$[\frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r} x_1'(s,\alpha)ds,$$

$$\frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r} x_2'(s,\alpha)ds]$$

So

$$[D_c^{\ r} x(t)]^\alpha = [D_c^{\ r} x_1(t,\alpha), D_c^{\ r} x_2(t,\alpha)]$$

Remark 4.1 When

$[f(t,x)]^\alpha = [f_1(t,x_1(t,\alpha),x_2(t,\alpha)), f_2(t,x_1(t,\alpha),x_2(t,\alpha))]$

then we translate the fuzzy fractional initial value problem (6) into a system of fractional initial value problems.

$$\begin{cases} D_c^{\ r} x_1(t,\alpha) = f_1(t,x_1(t,\alpha),x_2(t,\alpha)), \\ x_1(t_0,\alpha) = x_1^{0,\alpha} \\ \\ D_c^{\ r} x_2(t,\alpha) = f_2(t,x_1(t,\alpha),x_2(t,\alpha)), \\ x_2(t_0,\alpha) = x_2^{0,\alpha} \end{cases} \qquad (7)$$

when part (i) of Theorem 4.1 holds and

$$\begin{cases} D_c^{\ r} x_1(t,\alpha) = f_2(t,x_1(t,\alpha),x_2(t,\alpha)), \\ x_1(t_0,\alpha) = x_1^{0,\alpha} \\ \\ D_c^{\ r} x_2(t,\alpha) = f_1(t,x_1(t,\alpha),x_2(t,\alpha)), \\ x_2(t_0,\alpha) = x_2^{0,\alpha} \end{cases} \qquad (8)$$

when part (ii) of Theorem 4.1 holds. In these equations $[x(t_0)]^\alpha = [x_1^{0,\alpha}, x_2^{0,\alpha}]$ is a fuzzy initial value.

In each case, the system of crisp fractional initial value problems will be solved by Adomian decomposition method. Finally, it must be verified whether the results make α -levels of a fuzzy number. When the result is a fuzzy number, by substituting in the (9) and using metric D the error of approximate solution will be obtained.

4. Conclusion

Fuzzy fractional differential equations have been introduced in Caputo sense. Adomian decomposition method has been applied to obtain approximate fuzzy solution. Although the method given in this paper is for the fuzzy fractional initial value problem, it might also be applicable to fuzzy fractional partial differential equations.

Acknowledgements:

Authors are grateful to the Islamic Azad University, Saveh Branch, for financial support to carry out this work.

***Corresponding Author:**
Dr. Abdorreza Panahi
Department of Mathematics
Islamic Azad University
Saveh Branch, Saveh, Iran
E-mail: Panahi53@gmail.com

References

1. Abbasbandy S, Panahi A, Rouhparvar H. Solving fuzzy differential inclusions using the LU-representation of fuzzy numbers. J. Sci. I.A.U. 2010;19(74/2):79-88.
2. Abbasbandy S, Viranloo TA, Pouso OL, Nieto JJ. Numerical methods for fuzzy differential inclusions. Computers and Mathematics with Applications. 2004;48:1633-1641.
3. Allahviranloo T, Ahmady E, Ahmady N. Nth-order fuzzy linear differential equations. Information Sciences. 2008;178:1309-1324.
4. Allahviranloo T, Ahmady N, Ahmady E. Numerical solution of fuzzy differential equations by predictor-corrector method. Information Sciences. 2007;177: 1633-1647.
5. Allahviranloo T, Kiani NA, Barkhordari M. Toward the existence and uniqueness of solution of second-order fuzzy differential equations. Information Sciences. 2009;179:1207-1215.
6. Allahviranloo T, Kiani NA, Motamedi N. Solving fuzzy differential equations by differential transformation method. Information Sciences. 2009; 179:956-966.
7. Allahviranloo T, Panahi A, Rouhparvar H. A computational method to find an approximate analytical solution for fuzzy differential equations. An. St. Univ. Ovidius Constanta. 2009;17:5-14.
8. Benchohra M, Darwish MA. Existence and uniqueness theorems for fuzzy integral equations of fractional order. Communications in Applied Analysis. 2008;12:13-22.
9. Buckley JJ, Feuring T. Fuzzy differential equations. Fuzzy Sets and Systems. 2000;110:43-54.
10. Goetschel R, Woxman W. Elementary fuzzy calculus. Fuzzy Sets and Systems. 1986;18:31-43.

11. Hashim I, Abdulaziz O, Momani S. Homotopy analysis method for fractional IVPs. Communications in Nonlinear Science and Numerical Simulation. 2009;14:674-684.
12. Kaleva O. A note on fuzzy differential equations. Nonlinear Analysis. 2006;64:895–900.
13. Kaleva O. Fuzzy differential equations. Fuzzy Sets and Systems. 1987;24:301-317.
14. Kaleva O. The Cauchy problem for fuzzy differential equations. Fuzzy Sets and Systems. 1990;35:389-396.
15. Luchko Y, Gorenflo R. The initial value problem for some fractional differential equations with Caputo derivative. Fachbereich Mathematik und Informatic, Berlin, 1998.
16. Podlubny I. Fractional differential equations. Academic Press, New York, 1999.
17. Puri ML, Ralescu DA. Differentials of fuzzy functions. J. Math. Anal. Appl. 1983;91:552-558.
18. Wazwaz AM. The modified decomposition method and Pade approximants for solving the Thomas-Fermi equation. Applied Mathematics and Computation. 1999;105:11-19.

11/03/2010