Application of variational iteration method for solving the nonlinear generalized Ito system

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Abstract: In this article, we implement relatively analytical technique called the variational iteration method (VIM)for solving nonlinear generalized Ito system. In this method, a correction functional is constructed by a general Lagrange multiplier. Two cases are given to illustrate the accuracy and effectiveness of the method .We compare our results with results obtained by exact solution. This Comparison reveals that the variational iteration method is very effective, convenient and easier to be implemented.

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1. Introduction

In this paper, we extend the application of the variatioal iteration method to find approximate solutions for nonlinear generalized Ito system. The variational iteration method, which proposed by Ji-Huan He [1-3], is considered to find analytic and approximate solutions of differential equations. It's effectively and easily used to solve some classes of nonlinear problems, Variational iteration method has been favorably applied to various kinds of nonlinear problems. The main property of the method is in its flexibility and ability to solve nonlinear equations accurately and conveniently. Major applications to nonlinear wave equation, nonlinear fractional differential equations, nonlinear oscillations and nonlinear problems arising in various engineering applications are surveyed. The flexibility and adaptation provided by the method have made the method a strong candidate for approximate analytical solutions.

2. Variational Iteration method

To illustrate the basic concepts of the Variatioal iteration method, we consider the differential equation in the formal form

Lu +Nu=g(x),

where L is a linear operator, N a nonlinear operator and g(x) an inhomogeneous term. According to VIM, we can construct a correctional functional as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\xi) + N\overline{u}_n(\xi)\}d\xi ,$$

where λ is a Lagrangeian multiplier [4], which can be determined by using variational theory, the subscript n denotes the n-th order approximation, and $\overline{u}_n(\xi)$ is considered as a restricted variational, i.e. $\overline{u}_n(\xi)$ =0.

3. Application

Consider the nonlinear generalized Ito system of partial differential equations [5]

$$u_t = v_x , \quad (1)$$
$$v_t = -2(v_{xxx} + 3\,u\,v_x + 3\,v\,u_x) - 12\,w\,w_x + 6p_x , \quad (2)$$
$$w_t = w_{xxx} + u\,w_x , \quad (3)$$
$$p_t = p_{xxx} + u\,p_x . \quad (4)$$

To illustrate the degree of accuracy to VIM, two cases of nonlinear generalized Ito system exact are discussed in details.

3.1. Nonlinear generalized Ito system case 1

In this case the analytical solution for system (1-4)

$$u(x,t) = a_0 - 2\,k^2\,m^2\,Sn^2(\xi) , \quad (5a)$$

$$v(x,t) = -\frac{1}{2k^2m^2}(3k^6m^2(1+m^2)^2 - 12k^4m^2(1+m^2)a_0 + 9k^2m^2a_0^2 - c_1^2) \quad +2k^4m^2(1+m^2) - 6k^2m^2a_0)Sn^2(\xi) \quad (5b)$$

$$w(x,t) = c_0 + c_1\,Sn(\xi) , \quad (5c)$$

$$p(x,t) = e_0 + c_0c_1\,Sn(\xi) . \quad (5d)$$

where a_0, k, m, c_0, c_1 and $e_0$are constant, and $\xi = k\,x + (-\,k^3\,(1+\,m^2\,)+3ka\,)\,t + \xi_0$, and ξ_0 is constant. we start with initial approximation $u_0 = u(x,0)$, $v_0 = v(x,0)$, $w_0 = w(x,0)$ and $p_0 = p(x,0)$ given by

$$u_0(x,t) = a_0 - 2\,k^2\,m^2\,Sn^2(k\,x + \xi_0) , \quad (6a)$$

$v_0(x,t) =$

$$-\frac{1}{2k^2m^2}(3k^6m^2(1+m^2)^2 - 12k^4m^2(1+m^2)a_0 + 9k^2m^2a_0^{\,2} - c_1^{\,2})Sn^2(kx+\xi_0)$$

$w_0(x,t) = c_0 + c_1\, Sn(kx+\xi_0)$, (6c)

$p_0(x,t) = e_0 + c_0c_1\, Sn(kx+\xi_0)$. (6d)

To solve the system (1-4) by means of variational iteration method, we can construct the correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1\{u_{nt} - \tilde{v}_{nx}\}d\tau\,, \quad (7a)$$

$$v_{n+1}(x,t) = v_n(x,t) + \left[\int_0^t \lambda_2\{v_{nt} + 2(v_{nxxx} + 3\widetilde{u_n v_{nx}} + 3\widetilde{v_n u_{nx}}) + 12\widetilde{w_n w_{nx}} + 6\tilde{p}_{nx})\}d\tau\right] \quad (7b)$$

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3\{w_{nt} - w_{nxxx} - 3\widetilde{u_n w_{nx}}\}d\tau\,, \quad (7c)$$

$$p_{n+1}(x,t) = p_n(x,t) + \int_0^t \lambda_3\{p_{nt} - p_{nxxx} - 3\widetilde{u_n p_{nx}}\}d\tau\,. \quad (7d)$$

where λ_1, λ_2, λ_3 and λ_4 are Lagrange multipliers are to determined, and $\tilde{v}_{nx}$, $\widetilde{u_n v_{nx}}$, $\widetilde{v_n u_{nx}}$, $\widetilde{w_n w_{nx}}$, $\tilde{p}_{nx}$, $\widetilde{u_n w_{nx}}$ and $\widetilde{u_n p_{nx}}$ are denotes restricted variations, **i.e.** $\delta\tilde{v}_{nx} = 0$, $\delta\widetilde{u_n v_{nx}} = 0$, $\delta\widetilde{v_n u_{nx}} = 0$, $\delta\widetilde{w_n w_{nx}} = 0$, $\delta\tilde{p}_{nx} = 0$, and $\delta\widetilde{u_n p_{nx}} = 0$. Its stationary conditions can be obtained as follows :

$1 + \lambda_1|_{\tau=t} = 0\,,\quad \lambda_1' = 0 \;\Rightarrow\; \lambda_1 = -1,$ (8a)

$1 + \lambda_2|_{\tau=t} = 0\,,\quad \lambda_2' = 0 \;\Rightarrow\; \lambda_2 = -1,$ (8b)

$1 + \lambda_3|_{\tau=t} = 0\,,\quad \lambda_3' = 0 \;\Rightarrow\; \lambda_3 = -1,$ (8c)

$1 + \lambda_4|_{\tau=t} = 0\,,\quad \lambda_4' = 0 \;\Rightarrow\; \lambda_4 = -1.$ (8d)

substituting (8a - 8d) in (7a - 7d), and the following variational iteration formula can be obtained

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_{nt} - \tilde{v}_{nx}\}d\tau\,, \quad (9a)$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \{v_{nt} + 2(v_{nxxx} + 3\widetilde{u_n v_{nx}} + 3\widetilde{v_n u_{nx}}) + 12\widetilde{w_n w_{nx}} + 6\tilde{p}_{nx})\}d\tau\,, \quad (9b)$$

$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t \{w_{nt} - w_{nxxx} - 3\widetilde{u_n w_{nx}}\}d\tau\,, \quad (9c)$$

$$p_{n+1}(x,t) = p_n(x,t) - \int_0^t \{p_{nt} - p_{nxxx} - 3\widetilde{u_n p_{nx}}\}d\tau\,. \quad (9d)$$

with $n \geq 0$ **.**

By the above iteration formulas (9a - 9d),we can obtain directly the order components as

$u_1(x,t) = a_0 + 4\, k(3a_0\, k^2m^2 + k^4m^2 + k^4m^4)\, t\, Cn(kx+\xi_0)\, Dn(kx+\xi_0)\, Sn(kx+\xi_0) - 2\, k^2m^2\, Sn^2(kx+\xi_0)$,

$v_1(x,t) = 1/2k^2m^2\,\{ c_1^{\,2} - 9\, a_0^{\,2}\, k^2\, m^2 + 12\, a_0\, k^4\, m^2 - 3\, k^6\, m^2 + 12\, a_0\, k^4\, m^4 - 6k^6\, m^4 - 3\, k^6\, m^6 + 64\, k^7\, m^5(-3\, a_0 + k^2\, (1+m^2))t Cn^3\, (kx+\xi_0)\, Dn\, (kx+\xi_0)\, Sn\, (kx+\xi_0) + 4k^4\, m^4(-3\, a_0 + k^2(1+m^2))Sn^2\, (kx+\xi_0)\; -8\, k^5\, m^4\, t\, Cn\, (kx+\xi_0)\, Dn\, (kx+\xi_0)\, Sn\, (kx+\xi_0)\, (9\, a_0^{\,2} - 30\, a_0\, k^2 + 9\, k^4 - 30\, a_0 k^2\, m^2 + 18\, k^4\, m^2 + 9\, k^4\, m^4 + 8(3\, a_0 k^2 - k^4(1+m^2))\, Dn^2\, (kx+\xi_0) - 8\, k^2\, m(-1+3\, m)(-3\, a_0 + k^2(1+m^2)Sn^2\, (kx+\xi_0))\}$

$w_1(x,t) = c_0 - c_1\, k^3\, m\, t\, Cn^3(kx+\xi_0)\, Dn\,(kx+\xi_0) + c_1\, Sn\,(kx+\xi_0) + c_1\, k\, t\, Cn\,(kx+\xi_0)\, Dn\,(kx+\xi_0)\, (3\, a_0 - k^2\, Dn^2(kx+\xi_0) + 2\, k^2(2-3\, m)\, m\, Sn^2(kx+\xi_0)$,

$p_1(x,t) = e_0 + 2\, c_0\, c_1\, Sn\,(kx+\xi_0) - 2c_0\, c_1\, k^3\, t\; Cn\,(kx+\xi_0)\, Dn\,(kx+\xi_0)\, (m\, Cn^2\,(kx+\xi_0) + Dn^2\,(kx+\xi_0) - 4m\, Sn^2\,(kx+\xi_0)) + 6\, c_0 c_1\, k\, t\; Cn\,(kx+\xi_0)\, Dn\,(kx+\xi_0)\, (a_0 - 2k^2m^2Sn^2\,(kx+\xi_0)$.

and so on, in the same manner using *Mathematica* Package, we can evaluate the numerical solutions to the rest components of iteration formulas (9a - 9d) with *n* th approximations ($n = 3$). The obtained numerical results are summarized in Table 1.

Table 1. Comparison of the exact and numerical solutions for Ito system

x	t	Error u	Error v	Error w	Error p
x= -200	0	0	-3.683E-12	0	0
	0.5	-2.6514E-11	3.6744E-10	4.5074E-08	9.0147E-08
	1	-1.8466E-10	2.9358E-09	1.2389E-07	2.4777E-07
	1.5	-5.2322E-10	9.9244E-09	2.3644E-07	4.7288E-07
	2	-9.9882E-10	2.352E-08	3.8273E-07	7.6547E-07
x= -100	0	0	3.683E-12	0	0
	0.5	4.3126E-11	-1.8335E-09	-5.0306E-08	7.181E-08
	1	3.6293E-10	-1.4574E-08	-6.8267E-08	1.2719E-07
	1.5	1.2867E-09	-4.9215E-08	-5.3864E-08	1.6631E-07
	2	3.1974E-09	-1.1666E-07	-7.078E-09	1.8931E-07
x= 0	0	0	-3.638E-12	0	0
	0.5	3.0864E-11	7.4125E-10	-8.8631E-08	-1.7726E-07
	1	1.8954E-10	5.9626E-09	-2.1134E-07	-4.2269E-07
	1.5	4.3654E-10	3.3138E-08	-3.6814E-07	-7.3627E-07
	2	5.4754E-10	7.854E-08	-5.5901E-07	-1.118E-06
x= 100	0	0	3.683E-12	0	0
	0.5	-6.7617E-11	1.2224E-09	4.5074E-08	9.0147E-08
	1	-5.1269E-10	9.8225E-09	1.2389E-07	2.4777E-07
	1.5	-1.6298E-09	-1.6298E-09	2.3644E-07	4.7288E-07
	2	-3.6218E-09	-3.6218E-09	3.8273E-07	7.6547E-07
x= 200	0	0	-3.683E-12	0	0
	0.5	7.4224E-11	-1.0587E-09	4.9735E-08	9.9471E-08
	1	6.2245E-10	-8.502E-09	6.5361E-08	1.3072E-07
	1.5	2.2033E-09	-2.8707E-08	4.6877E-08	9.3753E-08
	2	5.4691E-09	-6.8048E-08	-5.7178E-09	-1.1436E-08

From these results we conclude that the variational iteration method for Ito system, gives high degree of accuracy in comparison with analytical solution (5a-5d). Now we study the diagrams obtained by VIM and analytical solutions to show the relation between u,v,w,p and x with different values for t, and the relation between u,v,w,p and t with different values for x.

3.2. Nonlinear generalized Ito system *case* 2

In this case the analytical solution for system (1-4)

$$u(x,t) = a_0 - 2k^2 m^2\, Cs^2(\xi), \quad (10a)$$

$$v(x,t) = -\frac{1}{2k^2}(3k^6(2-m^2)^2 - 12k^4(2-m^2)a_0 + 9k^2 a_0^2 - c_1^2) \quad (10b)$$
$$+2k^4(2-m^2) - 6k^2 a_0)Cs^2(\xi)$$

$$w(x,t) = c_0 + c_1\, Cs(\xi), \quad (10c)$$

$$p(x,t) = e_0 + c_0 c_1\, Cs(\xi). \quad (10d)$$

where a_0, k, m, c_0, c_1 and e_0 are constant, and $\xi = kx + (-k^3(1+m^2) + 3ka)t + \xi_0$, and ξ_0 is constant. we start with initial approximation $u_0 = u(x,0)$, $v_0 = v(x,0)$, $w_0 = w(x,0)$ and $p_0 = p(x,0)$ given by

$$u_0(x,t) = a_0 - 2k^2 m^2\, Sn^2(kx + \xi_0), \quad (11a)$$

$$v_0(x,t) = -\frac{1}{2k^2}(3k^6(2-m^2)^2 - 12k^4(2-m^2)a_0 + 9k^2 a_0^2 - c_1^2) \quad (11b)$$
$$+2k^4(2-m^2) - 6k^2 a_0)Cs^2(kx + \xi_0)$$

$$w_0(x,t) = c_0 + c_1\, Sn(kx + \xi_0), \quad (11c)$$

$$p_0(x,t) = e_0 + c_0 c_1\, Sn(kx + \xi_0). \quad (11d)$$

By using iteration formulas (9a - 9d),we can obtain directly case 2 order components as

$$u_1(x,t) = a_0 - 2k^2 Cs^2(kx + \xi_0) + 4k^3(3a_0 - k^2(-2+m^2))\, t\, Cs(kx + \xi_0)\, Ds(kx + \xi_0)\, Ns(kx + \xi_0).$$

$$v_1(x,t) = 1/(2k^2)\{c_1^2 - 3(3a_0^2 k^2 - 4a_0 k^4(-2+m^2) + k^6(-2+m^2)^2) - 4(3a_0 k^4 - k^6(-2+m^2))\, Cs^2(kx + \xi_0) +$$

$128\ k^7(3\ a_0 - k^2(-2+m^2)\ t\ Cs^3(k\ x + \xi_0)Ds(k\ x + \xi_0)Ns(k\,x + \xi_0) + 8\,k^5\,t\,Cs(k\,x + \xi_0)Ds(k\,x + \xi_0)$

$Ns(k\,x + \xi_0)\,(9\,a_0^{\,2} + 60\,a_0\,k^2 + 36\,k^4 - 30\,a_0\,k^2\,m^2 - 36\,k^4\,m^2 + 9\,k^4m^4 -$

$8(3\,a_0\,k^2 - k^4(-2 + m^2)Ds^2(k\,x + \xi_0) - 8(3\,a_0\,k^2 - k^4(-2 + m^2)Ns^2(k\,x + \xi_0)\}$

$w_1(x,t) = c_0 + c_1\,Cs(k\,x + \xi_0) + 2\,c_1\,k^3\,t\,Cs^2(k\,x + \xi_0)Ds(k\,x + \xi_0)\,Ns(k\,x + \xi_0) - c_1\,k^3\,t\,Ds^3(k\,x + \xi_0)$

$Ns(k\,x + \xi_0) - c_1\,k\,t\,Ds(k\,x + \xi_0)\,Ns(k\,x + \xi_0)\,(3\,a_0 + k2\,Ns^2(k\,x + \xi_0))$

$p_1(x,t) = e_0 + 2\,c_0\,c_1Cs(k\,x + \xi_0) + 4\,c_0\,c_1\,k^3t\,Cs^2(k\,x + \xi_0)Ds(k\,x + \xi_0)\,Ns(k\,x + \xi_0) -$

$2\,c_0\,c_1\,k^3\,t\,Ds^3(k\,x + \xi_0)\,Ns(k\,x + \xi_0) - 2\,c_0\,c_1\,k\,t\,Ds(k\,x + \xi_0)$

$Ns(k\ x + \xi_0)(3\ a_0 + k^2\ Ns^2(k\ x + \xi_0))$

and so on, in the same manner we can evaluate the rest components of iteration formulas (9a - 9d) with n th approximations (n =3).The obtained numerical results are summarized in Table 2. The behavior of the solution obtained by VIM and analytic solution are shown in Figs. (3a - 3d) and (4a - 4d). Now we study the diagrams obtained by VIM and analytical solutions to show the relation between u,v,w,p and x with different values for t, and the relation between u,v,w,p and t with different values for x. The behavior of the 3rd iteration obtained by VIM Fig[4a - 4d] and the analytic solution Fig[3a - 3d]. Then, the surfaces respectively show the solution u(x,t),v(x,t), w(x,t), p(x,t).

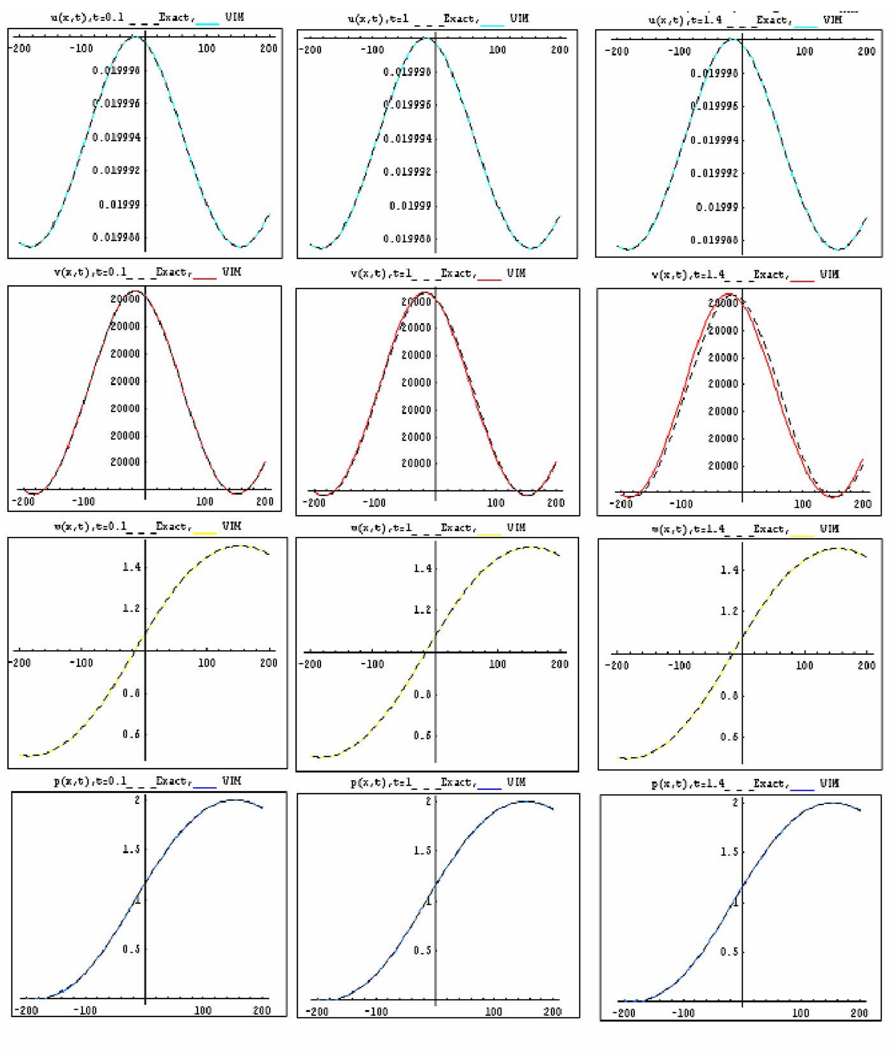


Fig show relation between u,v,w,p and x with constant values for t.

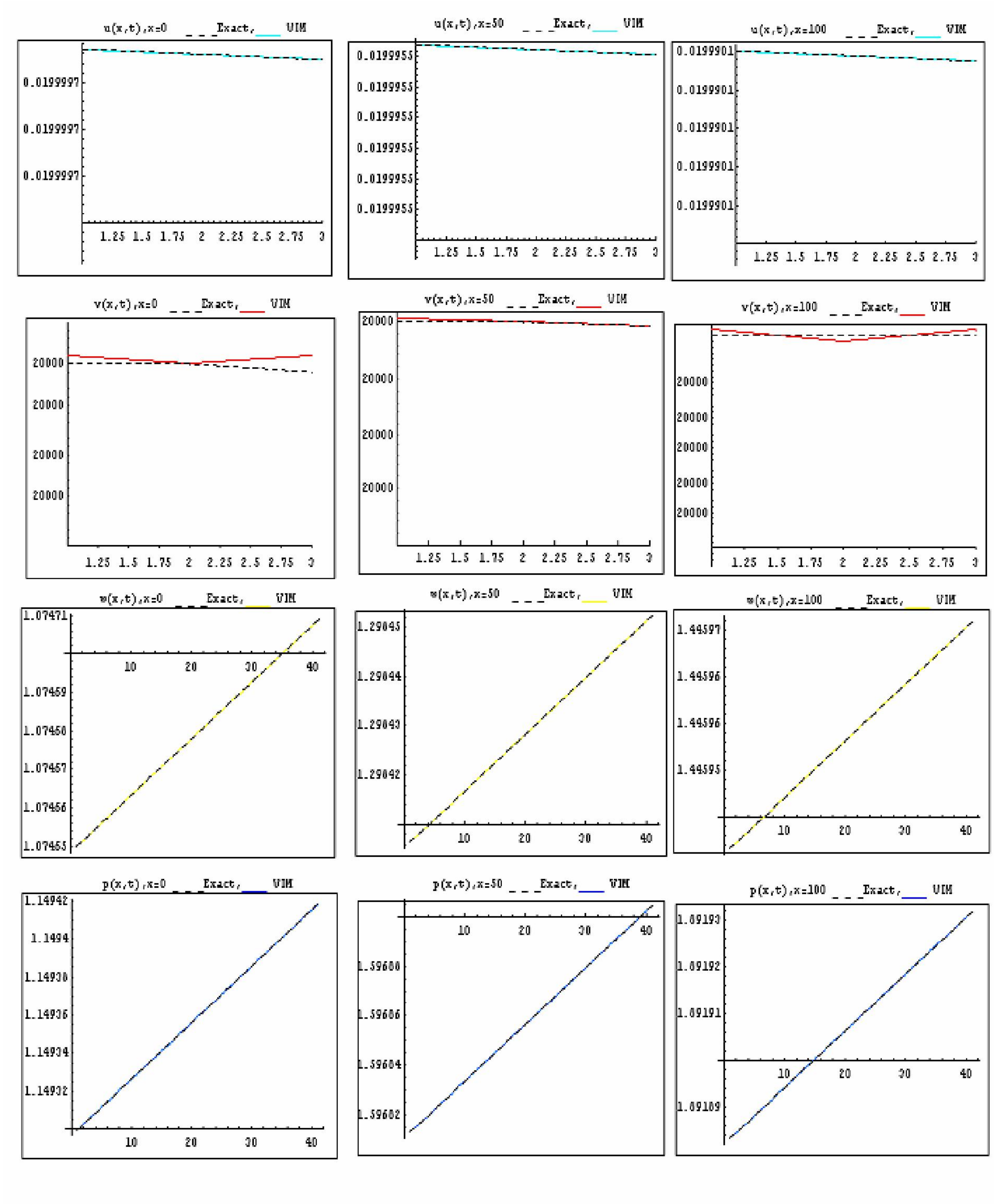
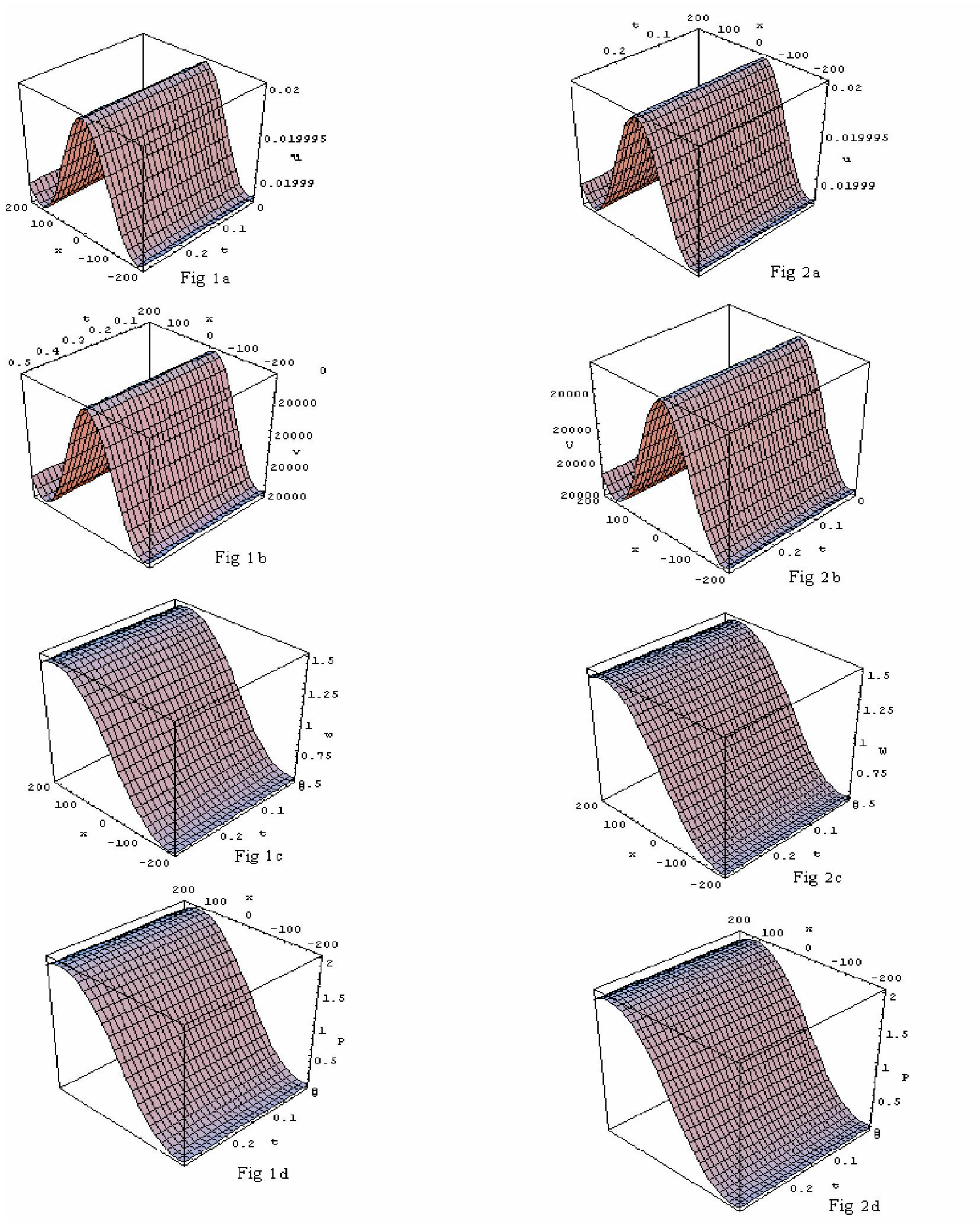


Fig show relation between u,v,w,p and t with constant values for x.



The behavior of the 3rd iteration obtained by VIM Fig[1a - 1d] and the analytic solution Fig[2a - 2d]. Then, the surfaces respectively show the solution u(x,t),v(x,t), w(x,t), p(x,t)

4- Conclusion

In this paper, the variational iteration method has been successfully used to find approximate solution for the nonlinear generalized Ito system of partial differential equation. The numerical results obtain using n approximations (n=3), compared with analytic solution show the high degree of accuracy.

Table 2. comparison of the 2nd case exact and numerical solutions for Ito system

	t	**Error** *u*	**Error** *v*	**Error** *w*	**Error** *p*
x= -200	*0*	0	0	0	0
	0.5	2.0061E-12	9.2723E-10	-3.5734E-08	-3.5734E-08
	1	1.593E-10	7.4278E-09	-5.9724E-08	-5.9724E-08
	1.5	8.869E-10	2.5063E-08	-7.1857E-08	-7.1857E-08
x= -120	*0*	0	0	0	0
	0.5	-5.0302E-11	-8.1702E-09	-9.4958E-08	-9.4958E-08
	1	2.7547E-10	-6.5469E-08	-2.614E-07	-2.614E-07
	1.5	2.7965E-09	-2.2111E-08	-4.9897E-07	-4.9897E-07
x= -40	*0*	0	0	0	0
	0.5	1.8093E-07	-8.5855E-06	-3.6379E-06	-3.6379E-06
	1	3.7522E-06	-0.000069122	-1.2928E-05	-1.2928E-05
	1.5	0.000020317	-0.00023475	-2.7729E-05	-2.7729E-05
x= 40	*0*	0	0	0	0
	0.5	3.6619E-09	1.7648E-07	1.0242E-07	1.0242E-07
	1	5.234E-08	1.4108E-06	7.5155E-07	7.5155E-07
	1.5	2.4907E-07	4.7559E-06	1.9522E-06	1.9522E-06
x= 120	*0*	0	0	0	0
	0.5	1.3132E-11	2.2646E-09	-3.17E-08	-3.17E-08
	1	3.1655E-10	-1.8146E-08	-3.6951E-08	-3.6951E-08
	1.5	1.6042E-09	6.1237E-08	-1.5605E-08	-1.5605E-08
x= 200	*0*	0	0	0	0
	0.5	-2.9265E-11	-3.9079E-09	-6.9876E-08	-6.9876E-08
	1	9.0988E-11	-3.1316E-08	-1.8075E-07	-1.8075E-07
	1.5	1.1602E-09	-1.0576E-07	-3.324E-07	-3.324E-07

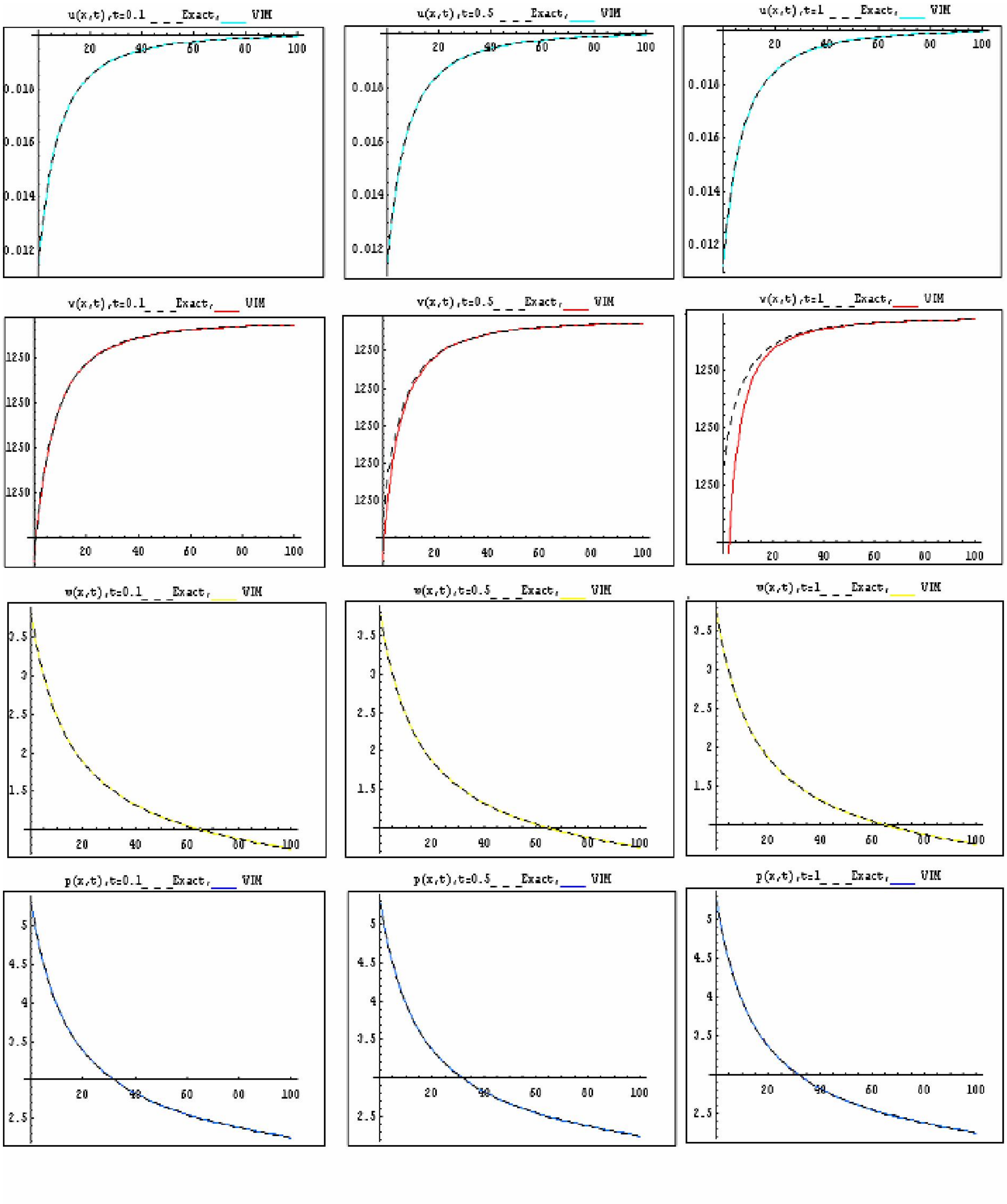


Fig show relation between u,v,w,p and x with constant values for t.

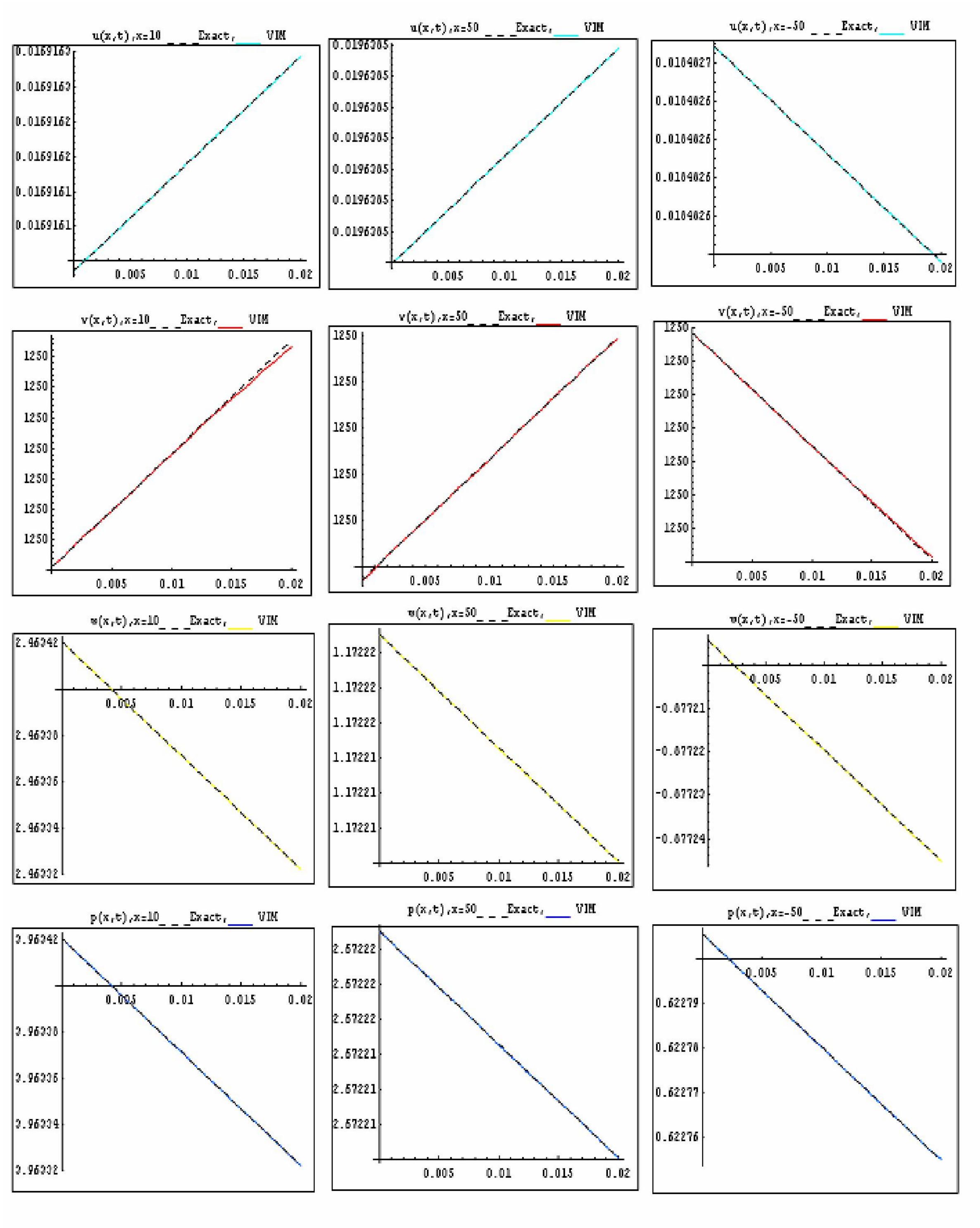
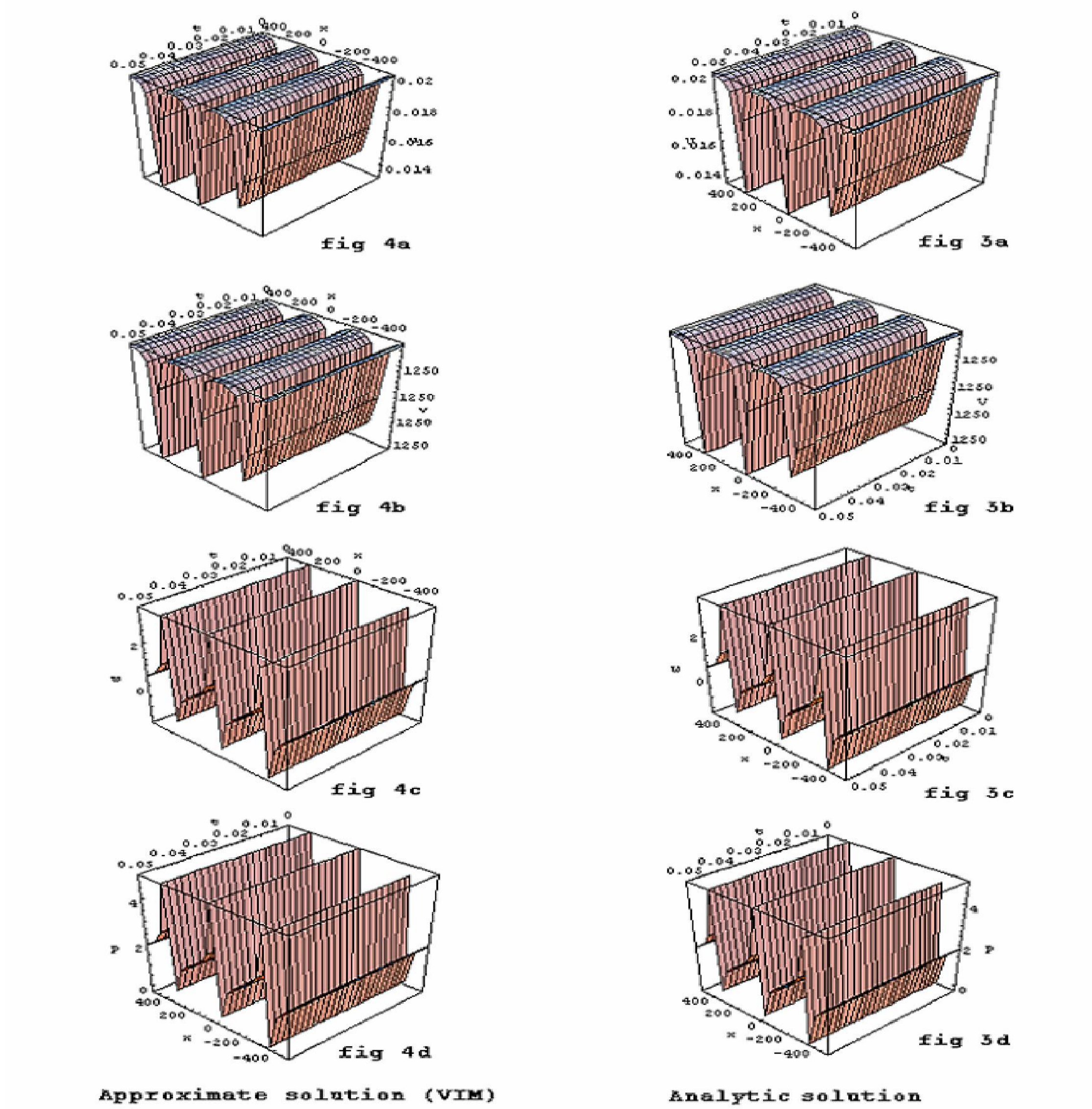


Fig show relation between u,v,w,p and t with constant values for x.



The behavior of the 3rd iteration obtained by VIM Fig[4a - 4d] and the analytic solution Fig[3a - 3d]. Then, the surfaces respectively show the solution u(x,t),v(x,t), w(x,t), p(x,t)

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