Structure and Some Geometric Properties of Nakano Difference Sequence Space

N. Faried and A.A. Bakery

Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt

awad_bakery@yahoo.com

Abstract: In this paper, we extend the class of sequence spaces of p-bounded variation bv_p introduced by Altay and Başar [Ukrainian Math.J.55(1)(2003),136-147]; where $1 \le p < \infty$ to the space of all sequences (x_k) such that $(x_k - x_{k-1})_{k=0}^{\infty}$ belongs to the sequence space introduced by Nakano $\ell_{(p_n)}$ where (p_n) is a sequence of positive numbers with $p_n \ge 1$,we define a modular functional on this space and show that when equipped with the Luxemburg norm is a Banach space and locally uniformly rotund when $p_n > 1 \, \forall \, n \in \mathbb{N}$, therefore possessing H-property and rotund . Finally we find Gurarii's modulus of convexity for the space bv_p . [Journal of American Science.2010;6(10):13-18]. (ISSN: 1545-1003).

Keywords: Gurarii's modulus of convexity, H-property, R-property, convex modular, Luxemburg norm, locally uniformly rotund.

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Introduction

Let (X,‖.‖) be a real Banach space and let B(X) (respe. S(X)) be the closed unit ball (resp. unit sphere) of X.

A point $x_0 \in S(X)$ is an H-point of B(X) if for any sequence (x_n) in X such that $\lim\limits_{n\to\infty} \| x_n \| = 1$, the weak convergence of x_n to x_0 $(write\ x_n \xrightarrow{W} x_0)$ implies that $\lim\limits_{n\to\infty} \| x_n - x_0 \| = 0$. If every point of S(X) is an H-point of B(X); then X is said to have H–property (Kadec-Klee).

A point $x \in S(X)$ is called an extreme point of B(X), if for any $y, z \in S(X)$, the equality x= $\frac{y+z}{2}$ implies y=z.

A Banach space X is said to be Rotund (R) if every point of S(X) is an extreme point of B(X). A point $x \in S(X)$ is called a locally uniformly rotund (LUR)-point, if for any sequence (x_n) in B(X) such that $\lim\limits_{n\to\infty} \| x_n + x \| = 2$, there holds that

$\lim\limits_{n\to\infty} \| x_n - x \| = 0$. If every point of S(X) is a LUR-point of B(X), then the space X is called locally uniformly rotund (LUR). It is known that if X is LUR, then it is rotund (R) and possesses property (H). However the converse of this last statement is not true in general. For these geometric notions and their role in mathematics we refer to the monographs [1], [2], [3], [4]. Some of these geometric properties were studied for orlicz spaces in [5], [6], [7], and [8]. By ω, we shall denote the space of all real or complex sequences and the set of natural numbers will denote by $\mathbb{N}$= {0,1,2,…}.

For a real vector space X over the real Numbers $\mathbb{R}$ a function $\sigma : X \to [0, \infty]$ is called a modular, if it satisfies the following conditions:

(i) $\sigma(x) = 0 \Leftrightarrow x = 0 \quad \forall x \in X$,

(ii) $\sigma(\lambda x) = \sigma(x)$, for all $\lambda \in \mathbb{R}$ with $|\lambda| = 1$,

(iii) $\sigma(\lambda x+\beta y)\le\sigma(x)+\sigma(y)$, $\forall x,y\in X$, $\forall\lambda,\beta\ge 0;\ \lambda+\beta=1$.

Further, the modular σ is called convex if

(iv) $\sigma(\lambda x+\beta y)\le\lambda\sigma(x)+\beta\sigma(y)\ \forall x,y\in X$ $\forall\lambda,\beta\ge 0;\ \lambda+\beta=1$.

Let $(X,\|.\|)$ be a normed linear space, consider Clarkson's modulus of convexity (Clarkson[10] and Day [11]) defined by

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2};x,y\in S(X),\|x-y\|=\varepsilon\right\},$$

where $0\le\varepsilon\le 2$.

The inequality $\delta_X(\varepsilon)>0$ for all $\varepsilon\in(0,2]$ characterizes the uniformly convex spaces. In [12], Gurarii's modulus of convexity is defined by

$$\beta_X(\varepsilon)=\inf\left\{1-\inf_{\alpha\in[0,1]}\|\alpha x+(1-\alpha)y\|;x,y\in S(X),\|x-y\|=\varepsilon\right\}$$

, where $0\le\varepsilon\le 2$. It is easily show that

$\delta_X(\varepsilon)\le\beta_X(\varepsilon)\le 2\delta_X(\varepsilon)$ for any $0\le\varepsilon\le 2$.

Also if $0<\beta_X(\varepsilon)<1$, Then X is uniformly convex, and if $\beta_X(\varepsilon)<1$, then X is strictly convex.

The Nakano sequence space $\ell(p)$ is defined by

$$\ell(p)=\{x=(x_k)\in\omega: m(tx)<\infty\ \mathit{for}\ \text{some t}>0\},$$

where $m(x)=\sum_{k=1}^{\infty}|x_k|^{p_k}$ and (p_k) is a sequence of positive real numbers with $p_k\ge 1$, $\forall\, k\in\mathbb{N}$.

The space $\ell(p)$ is a Banach space with the norm $\|x\|=\inf\left\{t>0:m(\frac{x}{t})\le 1\right\}$.

If $p=(p_k)$ is bounded, we can simply write

$$\ell(p)=\left\{x\in\omega:\sum_{k=1}^{\infty}|x_k|^{P_k}<\infty\right\}.$$

Also, some geometric properties of $\ell(p)$ were studied in [1], [3].

The idea of difference sequence was first introduced by Kizmaz [13]. Write $\Delta x_k=x_k-x_{k-1}$ for all $k\in\mathbb{N}$ with $x_{-1}=0$ and $\Delta:\omega\to\omega$ be the difference operator defined by

$$\Delta x=(x_k-x_{k-1})_{k=0}^{\infty}.$$

Definition: Let (p_n) be a sequence of positive real numbers with $p_n\ge 1$, we define the following sequence space

$$\ell_\Delta(p)=\{\mathrm{x}\in\omega:\rho(\lambda\mathrm{x})<\infty,\ \text{for some}\ \lambda>0\}$$

, where $\rho(x)=\sum_{k=0}^{\infty}|\Delta x_k|^{P_k}$ with the

norm $\|x\|=\inf\left\{t>0:\rho(\frac{x}{t})\le 1\right\}$.

If (p_n) is bounded, we can simply write

$\ell_\Delta(p)=\{x\in\omega:\sum_{k=0}^{\infty}|\Delta x_k|^{P_k}<\infty\}$. The space $\ell_\Delta(p)$ is a paranormed space by the paranorm

$$g_\Delta(x)=\left[\sum_{n=0}^{\infty}(|\Delta x_n|)^{p_n}\right]^{\frac{1}{H}}$$, see [16],

where $H=\sup_r p_r$. Throughout this paper, the sequence (p_n) is considered to be bounded

with $p_n>1\ \forall\ n\in\mathbb{N}$, and let $\sup_r p_r=H$. For any bounded sequence of positive numbers (p_k), we have

$|a_k+b_k|^{p_k}\le 2^{H-1}(|a_k|^{p_k}+|b_k|^{p_k})$, where $p_k\ge 1$ for all $k\in\mathbb{N}$.

Taking $P_n=P\ \forall n\in\mathbb{N}$, then $\ell_\Delta(p)=bv_p$, see [9]

Lemma 1:

The functional σ is convex modular on $\ell_\Delta(p)$.

Proof: It can be proved with standard techniques in a similar way as in [14, 15]

Lemma 2:

For $x \in \ell_\Delta(p)$, the modular σ on $\ell_\Delta(p)$ satisfies the following properties:

(i) If 0<r<1, then $r^H \sigma\left(\frac{x}{r}\right) \le \sigma(x)$ and $\sigma(rx) \le r\sigma(x)$.

(ii) If r>1, then $\sigma(x) \le r^H \sigma\left(\frac{x}{r}\right)$.

(iii) If r≥1, then $\sigma(x) \le r\sigma(x) \le \sigma(rx)$.

Proof: It can be proved with standard techniques in a similar way as in [14, 15]

Lemma 3:

For any, $x \in \ell_\Delta(p)$ the following assertions are satisfied:

(i) If $\| x \| < 1$, then $\sigma(x) \le \| x \|$.

(ii) If $\| x \| > 1$, then $\sigma(x) \ge \| x \|$.

(iii) $\| x \| = 1$ If and only if $\sigma(x) = 1$.

(iv) If $0 < r < 1$ and $\| x \| > r$, then $\sigma(x) > r^H$.

(v) If $r \ge 1$ and $\| x \| < r$, then $\sigma(x) < r^H$.

Proof : It can be proved with standard techniques in a similar way as in [14,15].

Lemma 4:

Let (x_n) be a sequence in $\ell_\Delta(p)$

(i) If $\lim_{n\to\infty} \| x_n \| = 1$, then $\lim_{n\to\infty} \sigma(x_n) = 1$,

(ii) if $\lim_{n\to\infty} \sigma(x_n) = 0$, then $\lim_{n\to\infty} \| x_n \| = 0$.

Proof :(i) Suppose that $\lim_{n\to\infty} \| x_n \| = 1$.Then for any $\varepsilon \in (0,1)$ there exists n_o such that $1-\varepsilon < \| x_n \| < 1+\varepsilon \ \forall\ n \ge n_0$. By lemma (3), $(1-\varepsilon)^H < \sigma(x_n) < (1+\varepsilon)^H$ implies that $\lim_{n\to\infty} \sigma(x_n) = 1$.

(ii) If $\lim_{n\to\infty} \| x_n \| \ne 0$, then there is an $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) such that $\| x_{n_k} \| > \varepsilon^H \ \forall\ k \in \mathbb{N}$.

This implies that $\lim_{n\to\infty} \sigma(x_{n_k}) \ne 0$ and hence $\lim_{n\to\infty} \sigma(x_n) \ne 0$.

Main results

Theorem1: $\ell_\Delta(p)$ is a Banach space with respect to Luxemburg norm defined by

$$\| x \| = \inf\left\{\rho > 0 : \sigma\left(\frac{x}{\rho}\right) \le 1\right\}.$$

Proof: Let $x_n = (x_n(k))_{k=1}^{\infty}, n = 0,1,2,...$ be a Cauchy sequence in $\ell_\Delta(p)$ is convergent according to the Luxemburg norm. Thus $\forall\, \varepsilon \in (0,1)\ \exists\ n_o$ such that $\| x_n - x_m \| < \varepsilon^H \ \forall\ m,n \ge n_0$. By Lemma 3(i) we obtain

$$\sigma(x_n - x_m) < \| x_n - x_m \| < \varepsilon^H \ \forall\ m,n \ge n_0.$$

(1) That is

$\sum_{r=0}^{\infty} | \Delta x_n(r) - \Delta x_m(r) |^{P_r} < \varepsilon^H \ \forall\, m,n \ge n_0$. For fixed r we get that

$| \Delta x_n(r) - \Delta x_m(r) | < \varepsilon \ \forall\, m,n \ge n_0$, and the sequence $(x_m(k))$ is a Cauchy sequence of real numbers. Let $x(k) = \lim_{m\to\infty} x_m(k)$, then

$\lim_{m\to\infty} \Delta x_m(r) = \Delta x(r)$. Therefore,

$\sum_{r=0}^{\infty} | \Delta x_n(r) - \Delta x(r) |^{P_r} < \varepsilon^H \ \forall\, n \ge n_0$. That is $\sigma(x_n - x) < \varepsilon^H \Rightarrow \lim_{n\to\infty} x_n = x$.

By the following calculations, we obtain that

$$\sum_{r=0}^{\infty} | \Delta x(r) |^{P_r} = \sum_{r=0}^{\infty} \left(| \Delta x(r) - \Delta x_n(r) | + | \Delta x_n(r) | \right)^{P_r}$$
$$\le 2^{H-1}\left[\sum_{r=0}^{\infty} | \Delta x(r) - \Delta x_n(r) |^{P_r} + \sum_{r=0}^{\infty} | \Delta x_n(r) |^{P_r} \right] < \varepsilon,$$

we see that the sequence x_n converges to $x = (x(k)) \in \ell_\Delta(p)$.This completes the proof of theorem.

Theorem2:

Let $(x_n) \subseteq B(\ell(p))$ and $(y_n) \subseteq B(\ell(p))$

If $\underset{n\to\infty}{Lim}\,\sigma\left(\frac{x_n + y_n}{2}\right) = 1$, then $\underset{n\to\infty}{Lim}\left(x_n(k) - y_n(k)\right) = 0$, for all $k \in \mathbb{N}$.

proof: See [15 Proposition2.6].

Theorem3: Let $(x_m) \subseteq B(\ell_\Delta(p))$

If $\underset{n\to\infty}{Lim}\,\sigma\left(\frac{x_n + x}{2}\right) = 1$, then $\underset{n\to\infty}{Lim}\, x_n(k) = x(k)$,$\forall\ k \in \mathbb{N}$.

Proof:

For each $k, m \in \mathbb{N}$, let $S_m(k) = \operatorname{sgn}(\Delta x_m(k) + \Delta x(k)\ if$ $\Delta x_m(k) + \Delta x(k) \neq 0$, and $S_m(k) = 1 if$ $\Delta x_m(k) + \Delta x(k) = 0$

, hence we have

$$1 = \underset{n\to\infty}{Lim}\,\sigma\left(\frac{x_m + x}{2}\right) = \sum_{k=0}^{\infty} \left| \frac{\Delta x_m(k) + \Delta x(k)}{2} \right|^{p_k} = \sum_{k=0}^{\infty}\left(S_m(k)\frac{\Delta x_m(k)}{2} + S_m(k)\frac{\Delta x(k)}{2}\right)^{P_n}$$

(2)

Let $\alpha_m(k) = S_m(k)\Delta x_m(k)$ and $\beta_m(k) = S_m(k)\Delta x(k)$, $\forall\ m,n \in \mathbb{N}$

Then $(\alpha_m),(\beta_m) \in \ell(p)$ and from (2) we have $\underset{m\to\infty}{Lim}\,\sigma\left(\frac{\alpha_m + \beta_m}{2}\right) = 1$.From Theorem (2), we have $\underset{m\to\infty}{Lim}\left(\alpha_m(k) - \beta_m(k)\right) = 0$ (3), for all $k \in \mathbb{N}$. Now we shall prove that $\underset{m\to\infty}{Lim}\, x_m(k) = x(k)$,$\forall\ k \in \mathbb{N}$. From (3) we have at $k = 0$ that $\underset{m\to\infty}{Lim}\left(s_m(0)\Delta x_m(0) - s_m(0)\Delta x(0)\right) = 0$ .This implies that $\underset{m\to\infty}{Lim}\,\Delta x_m(0) = \Delta x(0)$.

Assume that $\underset{m\to\infty}{Lim}\,\Delta x_m(k) = \Delta x(k)$, $\forall\, k \leq n-1$.

Then we get

$$\underset{m\to\infty}{Lim}\left(s_m(k)(\Delta x_m(k) - \Delta x(k))\right) = 0\ \forall\, k \leq n-1,$$

since

$$s_m(n)(\Delta x_m(n) - \Delta x(n)) = \left(\alpha_m(n) - \beta_m(n)\right) - \sum_{k=0}^{n-1} s_m(k)(\Delta x_m(k) - \Delta x(k))$$

. (4)

It follows that from (3) and (4) that $s_m(n)(\Delta x_m(n) - \Delta x(n)) \to 0$ as $m \to \infty$.

This implies $\underset{m\to\infty}{Lim}\,\Delta x_m(k) = \Delta x(k)\ \forall\ k \in \mathbb{N}$, also we get $\underset{m\to\infty}{Lim}\, x_m(k) = x(k)$.

Theorem4: The space $\ell_\Delta(p)$ is LUR.

Proof: Let $(x_n) \subseteq B(\ell_\Delta(p))$ and $x = (x(r))_{r=0}^{\infty} \in S(\ell_\Delta(p))$ be such that $\underset{n\to\infty}{Lim} \| \frac{x_n + x}{2} \| \to 1$, by lemma (4-i) we have $\underset{n\to\infty}{Lim}\,\sigma(\frac{x_n + x}{2}) = 1$, by theorem (3) we have $\underset{n\to\infty}{Lim}\, x_n(i) = x(i)$, $\forall\, i \in \mathbb{N}$, for $\varepsilon > 0$ since $\sigma(x) = \sum_{r=0}^{\infty} (|\,\Delta x(r)\,|)^{P_r} < \infty$, $r_0 \in \mathbb{N}$ such that

$$\sum_{r=r_0+1}^{\infty} (|\,\Delta x(r)\,|)^{P_r} < \frac{\varepsilon}{3(2^{H+1})} \quad .$$

(5)

Since

$$\underset{n\to\infty}{Lim}\left(\sigma(x_n) - \sum_{r=0}^{r_0}(|\,\Delta x_n(r)\,|)^{P_r}\right) = \sigma(x) - \sum_{r=0}^{r_0}(|\,\Delta x(r)\,|)^{P_r}$$

, and $\underset{n\to\infty}{Lim}\,\Delta x_n(r) = \Delta x(r)$,

$\forall\, r \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$,

such that

$$\left| \sum_{r=r_0+1}^{\infty}(|\,\Delta x_n(r)\,|)^{P_r} - \sum_{r=r_0+1}^{\infty}(|\,\Delta x(r)\,|)^{P_r} \right| < \frac{\varepsilon}{3(2^H)}$$

(6)

$\forall\ n \geq n_0$, since $\underset{n\to\infty}{Lim}\, x_n(r) = x(r)$ .Since Δ is continuous operator we get

$|\Delta x_n(r) - \Delta x(r)| < \varepsilon, \forall\ n \geq n_0$.As a result

$\forall\, n \geq n_0$, we have

$$\sum_{r=0}^{r_0} (|\Delta x_n(r) - \Delta x(r)|)^{P_r} < \frac{\varepsilon}{3}$$

(7). Then, from (5), (6) and (7) it follows

that $\forall\, n \geq n_0$, we have

$$\sigma(x_n - x) = \sum_{r=0}^{\infty} |\Delta x_n(r) - \Delta x(r)|^{P_r} =$$

$$\sum_{r=0}^{r_0} (|\Delta x_n(r) - \Delta x(r)|)^{P_r} + \sum_{r=r_0+1}^{\infty} (|\Delta x_n(r) - \Delta x(r)|)^{P_r}$$

$$< \frac{\varepsilon}{3} + 2^H \left[\sum_{r=r_0+1}^{\infty} |\Delta x(r)|^{P_r} + \sum_{r=r_0+1}^{\infty} |\Delta x_n(r)|^{P_r} \right]$$

$$= \frac{\varepsilon}{3} + 2^H \left[2 \sum_{r=r_0+1}^{\infty} |\Delta x(r)|^{P_r} + \frac{\varepsilon}{3(2^M)} \right] <$$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, this shows that

$\underset{n\to\infty}{Lim}\, \sigma(x_n - x) = 0$ .Hence by lemma 4 (ii), we have $\underset{n\to\infty}{Lim} \| x_n - x \| \to 0$.

Theorem5: Guarii's modulus of convexity for the normed space

bv_p, $p \geq 1$ is $\beta_{bv_p}(\varepsilon) \leq 1 - (1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}$ where $0 \leq \varepsilon \leq 2$.

Proof: For $x \in bv_p$, we get that

$\| x \|_{bv_p} = \| \Delta x \|_{\ell_p} = (\sum_k |\Delta x_k|^p)^{\frac{1}{p}}$, and

$0 \leq \varepsilon \leq 2$, Consider the two Sequences

$x = (\Delta^{-1}(1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}, \Delta^{-1}(\frac{\varepsilon}{2}), 0,0,0,...)$ and

$y = (\Delta^{-1}(1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}, \Delta^{-1}(\frac{-\varepsilon}{2}), 0,0,0,...)$, we get that

$\Delta x = ((1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}, (\frac{\varepsilon}{2}), 0,0,0,...)$, and

$\Delta y = ((1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}, (\frac{-\varepsilon}{2}), 0,0,0,...)$.

Since $\| x \|_{bv_p}^p = \| \Delta x \|_{\ell_p}^p = 1 - (\frac{\varepsilon}{2})^p + (\frac{\varepsilon}{2})^p = 1$,

then $x \in S(bv_p)$ and

$\| y \|_{bv_p}^p = \| \Delta y \|_{\ell_p}^p = 1 - (\frac{\varepsilon}{2})^p + |(\frac{-\varepsilon}{2})|^p = 1$, then $y \in S(bv_p)$

and $\| x - y \|_{bv_p} = \| \Delta x - \Delta y \|_{\ell_p} = \varepsilon$, since

$\inf_{0\leq\alpha\leq 1} \| \alpha x + (1-\alpha) y \|_{bv_p} =$

$\inf_{0\leq\alpha\leq 1} \| \alpha \Delta x + (1-\alpha)\Delta y \|_{\ell_p} =$

$$\inf_{0\leq\alpha\leq 1} [|\alpha(1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}} + (1-\alpha)(1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}|^p + |\alpha(\frac{\varepsilon}{2}) + (1-\alpha)(-\frac{\varepsilon}{2})|^p]^{\frac{1}{p}}$$

$$= \inf_{0\leq\alpha\leq 1} [(1 - (\frac{\varepsilon}{2})^p) + |2\alpha - 1|^p\, (\frac{\varepsilon}{2})^p]^{\frac{1}{p}}$$

$$= [1 - (\frac{\varepsilon}{2})^p]^{\frac{1}{p}},$$

Consequently for $p \geq 1$, we get

$$\beta_{bv_p}(\varepsilon) \leq 1 - (1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}.$$

Corollary

(i) If $\varepsilon = 2$, then $\beta_{bv_p} \leq 1$, hence β_{bv_p} is strictly convex,

(ii) if $0 < \varepsilon < 2$, then $0 < \beta_{bv_p}(\varepsilon) \leq 1$, and hence bv_p is uniformly convex,

(iii) if $\alpha = \frac{1}{2}$, then $\delta_{bv_p}(\varepsilon) = \beta_{bv_p}(\varepsilon)$.

Corresponding author
N. Faried
Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
n_faried@hotmail.com

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